

INTRODUCTION

Numerical homotopy continuation is used to find the zero set of a difficult polynomial, $f(z)$, by starting at solutions of an easy problem $g(z) = 0$, and deforming $g(z)$ until solutions $f(z) = 0$ are obtained. The polynomials are connected by the equation

$$H(t, z) = t\gamma g(z) + (1 - t)f(z) \quad (1)$$

where $z \in \mathbb{C}$, with real-valued $t \in [0, 1]$. The complex constant $\gamma = e^{i\theta}$ with $\theta \in (-\pi, \pi)$ ensures the homotopy $H(t, z) \in \mathbb{C}$.

SOLUTION METHOD

Rewriting (1) as an initial value problem:

$$\frac{dz(t)}{dt} = -\frac{\gamma g(z(t)) - f(z(t))}{t\gamma g'(z(t)) + (1 - t)f'(z(t))}, \quad t \in [0, 1] \quad (2)$$

$$z(t_0) = z_0, \quad t_0 = 1 \quad (3)$$

we use a predictor-corrector method, $\Phi^{(k)}(z)$, that combines Euler's method with Newton's method to solve the IVP. As $t : 1 \rightarrow 0$, if k large enough then $\Phi^{(k)}(z)$ reaches a fixed point z^* and $f(z^*) = 0$.

CHARACTERIZING ERROR

When solving univariate polynomials, the following errors may occur:

False Multiplicities. Caused by “path jumping”: when two paths converge with each other, one root target will be hit twice and one will be missed. See paths 1 and 3 in figure 1.

Spurious Roots. Caused by ill-conditioned roots: root targets are not reached by final k in $\Phi^{(k)}(z) = z$ and $z \neq z^*$. See paths 8, 12, 16 in figure 1.

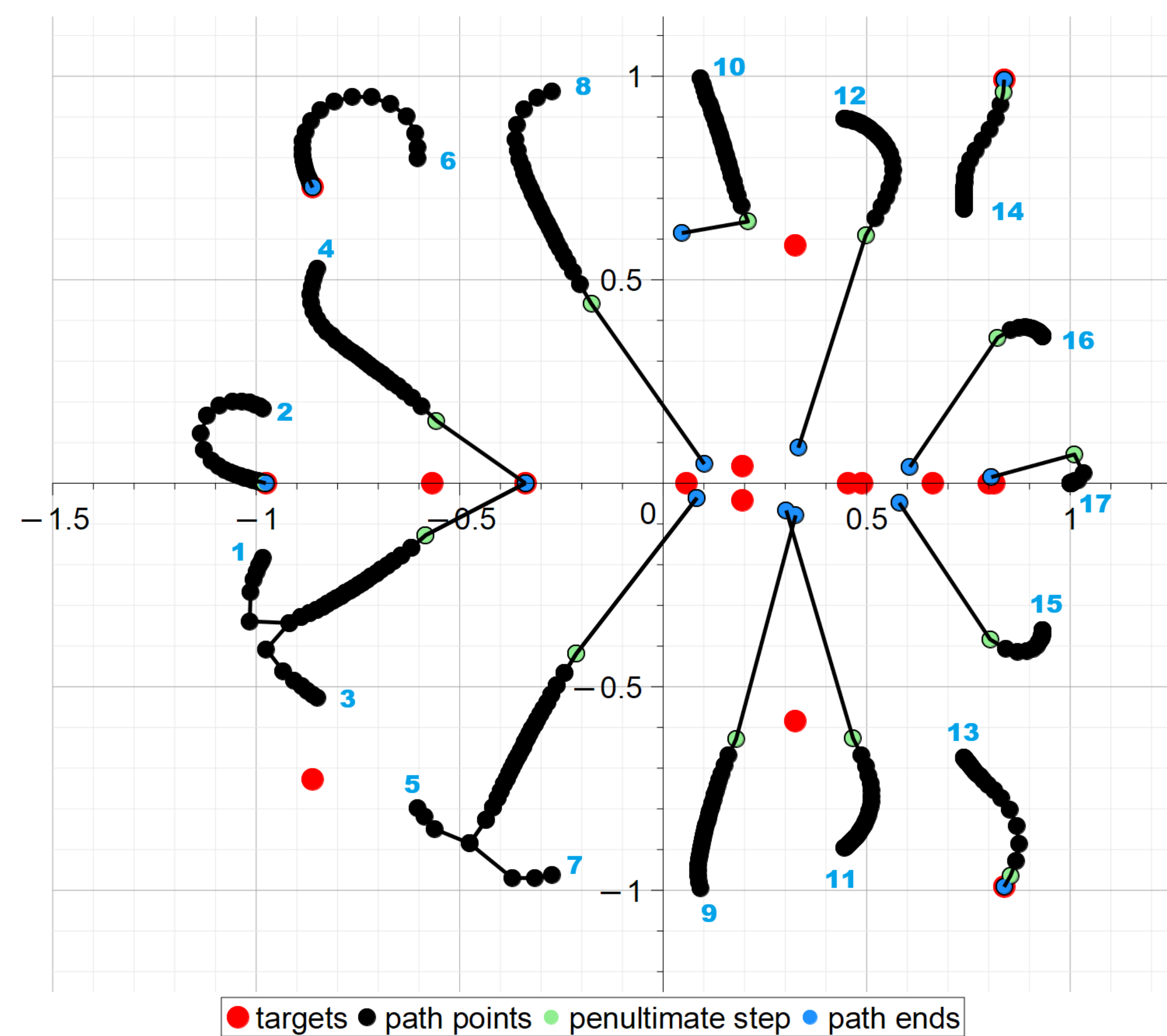


Figure 1. Homotopy paths leading to the roots of $f(z)$, a degree 17 polynomial with nine real roots. 30 Euler steps and 7 Newton iterations are used.

FALSE MULTIPLICITIES

Detection. For any two solution paths $\vec{z}_\alpha$, $\vec{z}_\beta$, and the same value of t , verify path jumping with the Lipschitz condition:

$$|H(t, \vec{z}_\alpha) - H(t, \vec{z}_\beta)| \leq q|\vec{z}_\alpha(t) - \vec{z}_\beta(t)|. \quad (4)$$

Resolution. Choose different γ , decrease the Euler step size $\Delta t = h$ by increasing k in predictor-corrector method, $\Phi^{(k)}(z)$.

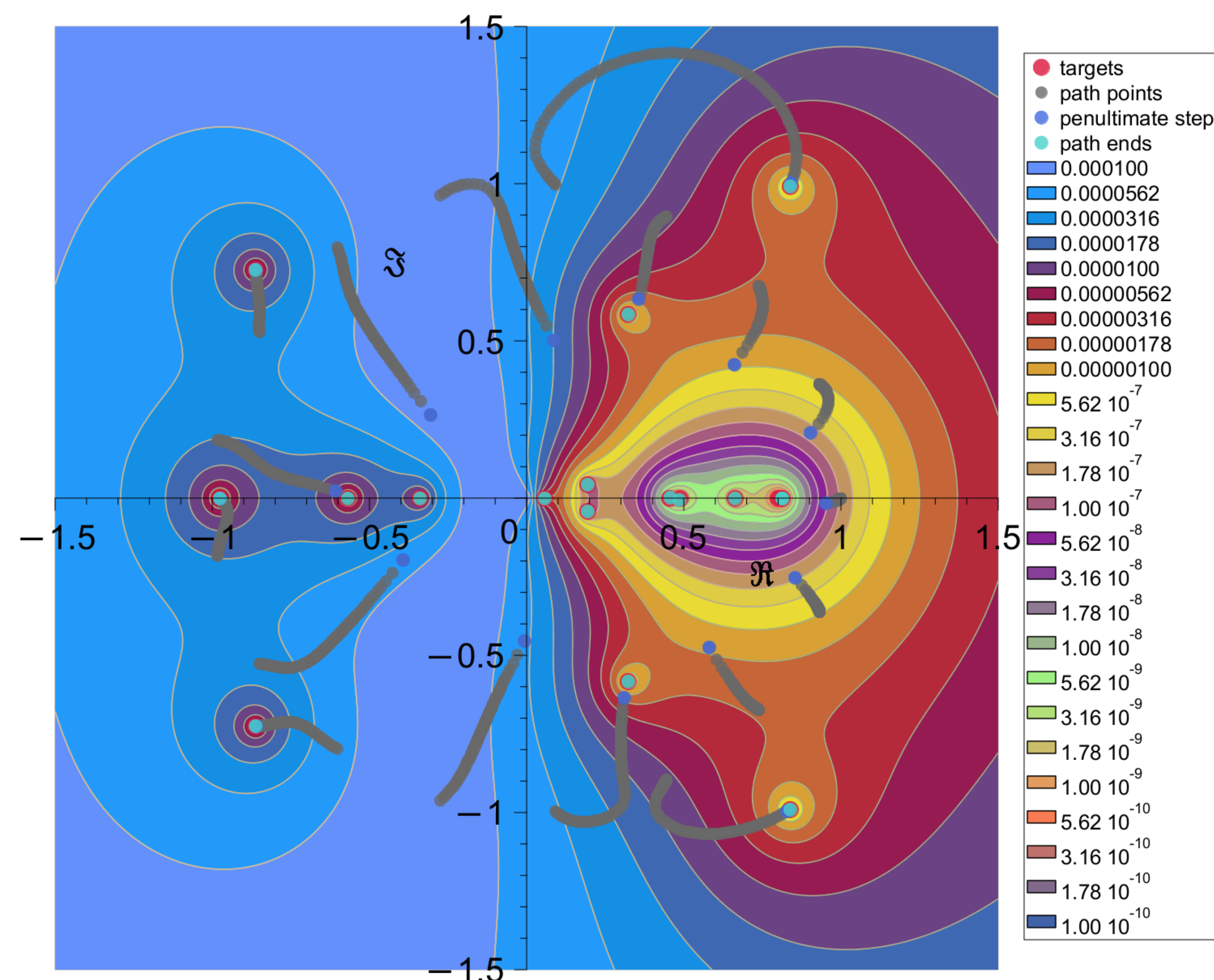


Figure 2. Pseudozero contours overlaid with homotopy paths leading to the roots of $f(z)$ expressed in the monomial basis. Path jumping is removed with different γ , more Euler steps. (80 Euler, 10 Newton.) Largest residual $|f(z)| < 10^{-6}$.

SPURIOUS ROOTS

Detection.

1. For some tolerance τ , check residuals $|f(z)| \leq \tau$.
2. Plot the pseudozero set of $f(z) = \sum_{k=0}^n c_k \varphi_k(z)$:

$$\Lambda_\epsilon(f) = \{z : (f + \Delta f)(z) = 0, \text{ where } |\Delta f| \leq \epsilon B(z)\}, \quad (5)$$

and $B(z) = \sum_{k=0}^n |c_k| |\varphi_k(z)|$ shows root conditioning without knowing the roots. Roots isolated only within small ϵ -contours given by $|f(z)|/B(z) \leq \epsilon$ may not be accessible with homotopy continuation in current polynomial basis.

Resolution.

1. Use Newton iteration taking the final output of $\Phi^{(k)}(z)$ for the z_0 in Newton's method.
2. Increase precision.
3. Change the polynomial basis.

CONCLUSION

False multiplicity is easily fixed with a change in the algorithm. Spurious roots may be addressed with adding another loop of Newton iterations to the algorithm; however, we may need to rewrite the polynomial into another basis.

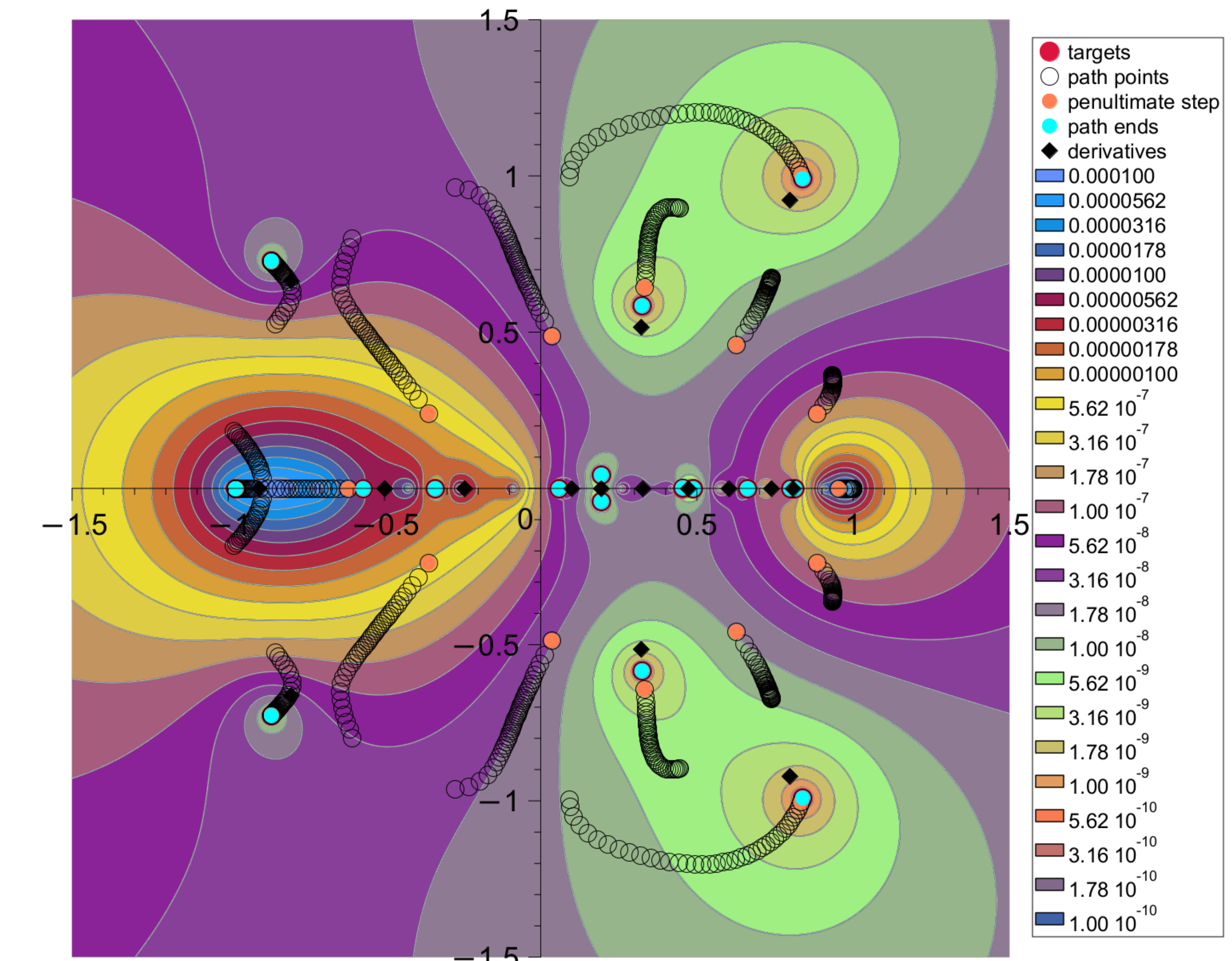


Figure 3. Pseudozero contours overlaid with homotopy paths leading to the roots of $f(z)$ expressed in the first barycentric form of the Lagrange basis (40 Euler steps, 7 Newton iterations + 25 Newton iterations; same γ as figure 2, with precision increased from 16 to 32 digits). Spurious roots removed, all root residuals are $|f(z)| < 10^{-10}$.

FUTURE WORK

The predictor-corrector method has faster run times in the monomial basis than the Lagrange basis, but more error in the results. We would like to see how other coefficient weights and other basis perform with homotopy continuation.

REFERENCES

1. Berrut, Jean-Paul and Trefethen, Lloyd N., (2004). Barycentric Lagrange Interpolation. SIAM Review. 26(3), 501–517.
2. Birkhoff, G. & Rota, G.C., (1989). Ordinary Differential Equations. Wiley& Sons.
3. Corless, Robert M., & Fillion, Nicolas (2013). A Graduate Introduction to Numerical Methods: From the Viewpoint of Backward Error Analysis. Springer.
4. Higham, Nicholas J. (2004). The Numerical Stability of Barycentric Lagrange Interpolation. IMA Journal of Numerical Analysis. 24(4), 547–556.
5. Mosier, Ronald G., (1986). Root Neighborhoods of a Polynomial. Mathematics of Computation, 47(175), 265–273.
6. Sommese, Andrew J. & Wampler II., Charles W., (2005). The Numerical Solution of Systems of Polynomials Arising in Engineering and Science. World Scientific.