# Applied Math 4615/9563 Assignment 3 Winter 2010

* problems are to be handed in on Tuesday February 23. Whenever possible use Maple to answer the following questions.

1. * Consider the pendulum of constant length $\ell$ where the location of the mass $m$ at time $t$ is $(x(t), y(t))$ which has constraint $x(t)^2 + y(t)^2 = \ell^2$ under constant gravity $g$. Notice the difficulty of having to differentiate with respect to a derivative - I have suggested a way to deal with this by using Maple's $D$ operator.

   (a) Define the Lagrangian $L := (x, y, xt, yt) \longrightarrow \frac{1}{2}m(xt^2 + yt^2) - mgy$.

   (b) Show that $L(x(t), y(t), \text{diff}(x(t), t), \text{diff}(y(t), t))$ gives you the correct result.

   (c) Now obtain the Euler-Lagrange (EL) equations [Hint: use the D operator, so that $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$ is given by $\text{diff}(D[3](L)(x(t), y(t), \text{diff}(x(t), t), \text{diff}(y(t), t)))$]

   (d) Substitute the polar coordinates $x(t) = r\sin(\theta(t))$, $y(t) = -r\cos(\theta(t))$ into $L$ to obtain $L$ in terms $\theta(t)$. Directly apply the EL equation with respect to $\theta$ with $L$ as in (a) and (b) to show that you obtain $\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0$.

   (e) Alternatively first form the Lagrangian equations of motion in $(x(t), y(t), \lambda(t))$ with the constraint $x(t)^2 + y(t)^2 = \ell^2$. Then substitute polar coordinates $x(t) = r\sin(\theta(t))$, $y(t) = -r\cos(\theta(t))$ into these equations. Explain how you can use these 3 equations to obtain the result in (d).

   (f) Briefly compare the difficulty of the methods (d) and (e) for this problem [verbal answer].

2. Consider the spherical pendulum of constant length $\ell$ where the location of the mass at time $t$ is $(x(t), y(t), z(t))$ which has constraint $x(t)^2 + y(t)^2 + z(t)^2 = \ell^2$ under constant gravity $g$. Note that you can find this problem in David Tong's notes.

   (a) Give the Lagrangian in terms of the $(x, y, z)$ coordinates.

   (b) Express the Lagrangian in terms of spherical coordinates $(\theta, \phi)$ where $x = \ell\cos(\phi)\sin(\theta)$, $y = \ell\sin(\phi)\sin(\theta)$, $z = -\ell\cos(\theta)$.

   (c) Use the Lagrangian from (a) to determine the EL equations the pendulum in terms of $(\theta, \phi)$.

3. Consider the 4 link pendulum shown in Figure 1, under constant gravity, with unit length links connecting masses of mass $m = m_1 = m_2 = m_3 = m_4$. The coordinates of each mass are $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $(x_4, y_4)$.

   (a) * Give the Lagrangian $L$ in terms of the $(x, y)$ coordinates.

   (b) * Give the constraints in terms of the $(x, y)$ coordinates.

   (c) * Give the constrained EL equations in terms of the $(x, y)$ coordinates and Lagrange multipliers.

   (d) * What alteration to the system in (b) and (c) should be made if it is connected as shown in Figure 2.

   (e) ‡Bonus challenge problem. Introduce a coordinate system that eliminates the constraints as much as possible for the system in Figure 2. Give the Lagrangian in the new coordinate system. The single link pendulum has one degree of freedom - how many degrees of freedom does the system in Figure 2 on the next page have?

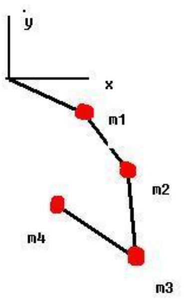


Fig 1

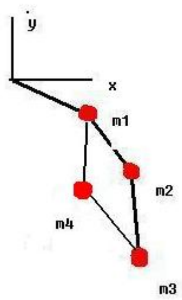


Fig 2

4. ♣ **Procedure for the n-dulum**. In this problem you are asked to make a procedure to produce the constrained EL (including the constraints) for an $n$ link n-dulum in 2 space dimensions $(x, y)$. That is the generalization of Figure 1, where the coordinates are $(x1, y1)$, $(x2, y2)$, ... , $(xn, yn)$ and the velocities are $(x1t, x2t)$, ... , $(xnt, ynt)$.

5. * **Gröbner Elimination (GE) is Great!** Consider the following systems of polynomial equations. In each case apply the command *Groebner[basis](sys, plex(vars));* where *vars* is the list of variables (e.g. for a two variable system it would be $vars := [x, y]$. To generate the systems you should exploit the command to generate a dense single polynomial of degree deg in the list of vars:
   *randpoly(vars, dense, degree = deg)*
   Recall that GE is a generalization of Gauss Elimination. Use experiments to make conjectures about:

   (a) Use *seq* to a make a procedure that returns *randsys(vars, deg, n)* - a random dense system of degree *deg* in the variables *vars* and $n$ equations.

   (b) What is the effect of GE on a system of random linear equations? Do some experiments.

   (c) What is the effect on a polynomial system of 1 equation in 1 variable?

   (d) What is the effect on a polynomial system of 2 equation in 2 variables?

   (e) What is the effect on a polynomial system of $n$ equation in $n$ variables?

   (f) In what sense is the system in triangular form? In what sense is it solved?