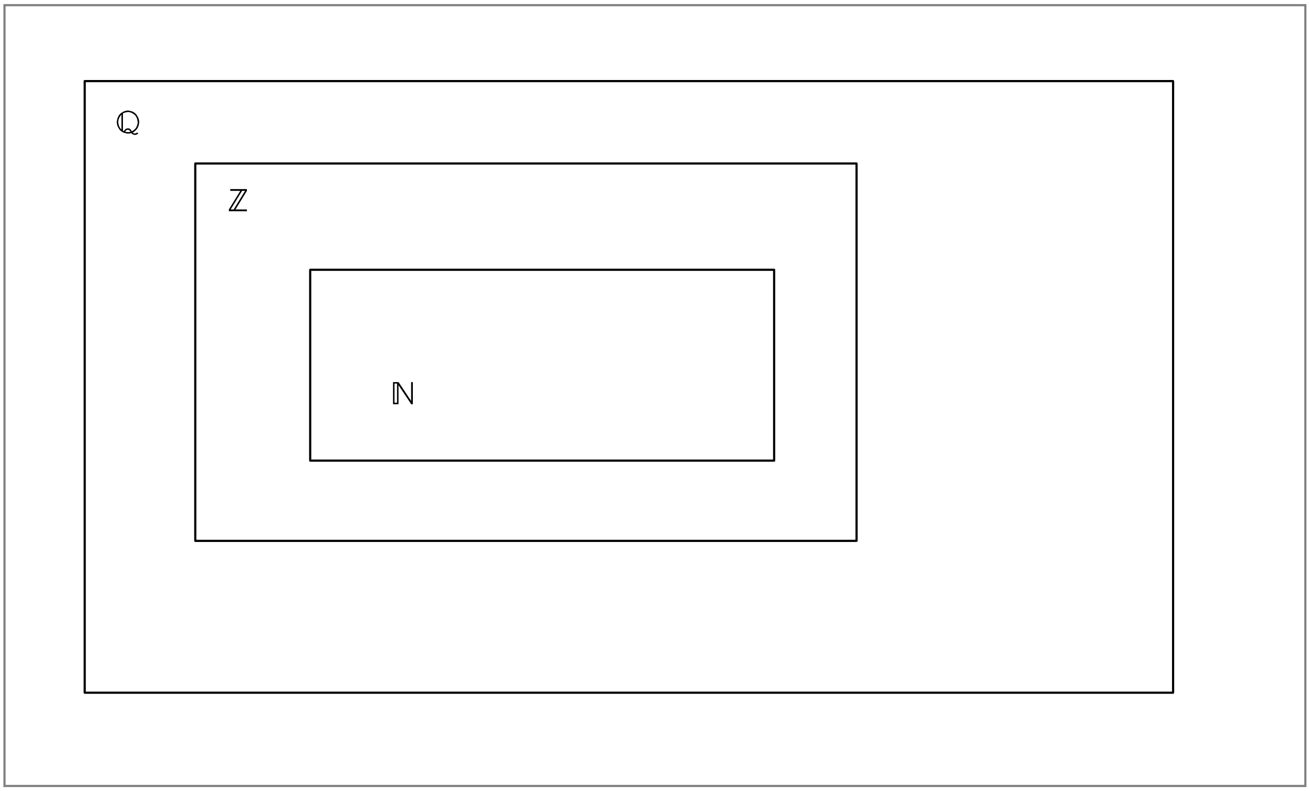
Recap from first lecture

The solution of equations phrased in each number system, motivated the extension of the number systems, to include those solutions.



Notice that $\mathbb{Z}$ is very nice and is the simplist familiar example of a ring:

Definition (Ring): A ring (R, +, *) is a set R, closed under two commutative binary operations + and * (ie. $a + b = b + a \in R$ **and** $a \cdot b = b \cdot a \in R$)

defining props of a ring

+ operations: closure (a + b in R); associative (a + (b + c) = (a + b) + c); there exists an additive identity 0 (a + 0 = a); there exists an additive inverse (a + (-a) = 0)
* operations: closure (a*b in R); associative (a*(b*c) = (a*b)*c); there exists a multiplicative identity 1 (1*a = a)
+/* interaction of +/* operations: a*(b+c) = a*b + a*c (distributive property).

As a side note: the + operations (CAIN) mean that (R, +) is a commutative group.

Notice that the rational numbers are particularly nice and is the simplist familiar example of a field.

Definition (Field): A field (R, +, *) is a ring (R, +, *) with $1 \neq 0$ which has one further important property:

most important defining prop of a field

Every nonzero element a in R has a multiplicative inverse b: a*b = 1. (you can call $b = \left(\frac{1}{a}\right)$)

For example any linear equation written with coeffients in $\mathbb{Q}$, has solutions in $\mathbb{Q}$ (why?).
Similarly if a system of linear equations with coefficients in $\mathbb{Q}$ has a unique solution, then this is also in $\mathbb{Q}$ (justify this).
Computer algebra systems love $\mathbb{Q}$.

Some books use the term commutative ring or field. We use the term just ring or field.

Now consider the problem (common in settlement of land claims in antiquity) of choosing a square with area 2.
What is the length of its sides? This problems is phrased as $x^2 = 2$ and even though it has coefficients in $\mathbb{Q}$, its
solution is not in $\mathbb{Q}$ (there is an easy proof that $\sqrt{2}$ is not rational due to the Greeks).