Buchberger's Algorithm

Input: $F = (f_1,\ldots,f_s)$
Output: a Groebner basis $G = (g_1,\ldots,g_t)$ for I, with $F \subset G$

$G := F$
REPEAT
  $G' := G$
  FOR each pair $\{p, q\}, p \neq q$ in G' **DO**
    $S := \overline{S(p,q)}^{G'}$
    IF $S \neq 0$ **THEN** $G := G \cup \{S\}$
  UNTIL $G = G'$

$with(Groebner):$
$A1 := 2\cdot x^2 + x\cdot y + x + 2\cdot y^2 + 3\cdot y + 3;$
$A2 := x^2 + x\cdot y + 3\cdot x - y^2 - 2\cdot y + 5;$
$Basis([A1, A2], plex(x, y));$

$$2x^2 + xy + x + 2y^2 + 3y + 3$$
$$x^2 + xy + 3x - y^2 - 2y + 5$$
$$[69 - 49y + 62y^2 + 83y^3 + 19y^4, 95 - 110y - 12y^2 + 19y^3 + 58x] \quad (1)$$

Choose a pair $\{A1, A2\}$.

$SA1A2 := \frac{2\cdot x^2}{2\cdot x^2}\cdot(A1) - \frac{2\cdot x^2}{x^2}\cdot(A2);$

$$-xy - 5x + 4y^2 + 7y - 7 \quad (2)$$

$\# \because SA1A2 = 0\cdot A1 + 0\cdot A2 + (-xy - 5x + 4y^2 + 7y - 7)$
$\# \therefore A3 := -xy - 5x + 4y^2 + 7y - 7$

$\mathbf{A3 := -x\cdot y - 5\cdot x + 4\cdot y^2 + 7\cdot y - 7;}$

$$-xy - 5x + 4y^2 + 7y - 7 \quad (3)$$

Choose a pair $\{A1, A3\}$.

$SA1A3 := expand\left(\frac{2\cdot x^2\cdot y}{2\cdot x^2}\cdot(A1) - \frac{2\cdot x^2\cdot y}{-x\cdot y}\cdot(A3)\right);$

$$9xy^2 + 15xy + 2y^3 + 3y^2 + 3y - 10x^2 - 14x \quad (4)$$

$SA1A3 := -10\cdot x^2 + 9xy^2 + 15xy - 14x + 2\cdot y^3 + 3\cdot y^2 + 3\cdot y:$

$\# \because SA1A3 = -5\cdot A1 + 0\cdot A2 + (-9y + 25)\cdot A3 + 2\cdot(58\cdot x + 19\cdot y^3 - 12y^2 - 110y + 95)$
$\# \therefore A4 := 58\cdot x + 19\cdot y^3 - 12y^2 - 110y + 95$

$A4 := 58\cdot x + 19\cdot y^3 - 12\,y^2 - 110\,y + 95$

$$95 - 110\,y - 12\,y^2 + 19\,y^3 + 58\,x \quad (5)$$

Choose a pair $\{A2, A3\}$.

$SA2A3 := expand\left(\frac{x^2\cdot y}{x^2}\cdot(A2) - \frac{x^2\cdot y}{-x\cdot y}\cdot(A3)\right);$

$$5\,x\,y^2 + 10\,x\,y - y^3 - 2\,y^2 + 5\,y - 5\,x^2 - 7\,x \quad (6)$$

$SA2A3 := -5\cdot x^2 + 5\cdot x\cdot y^2 + 10\cdot x\cdot y - 7\cdot x - y^3 - 2\cdot y^2 + 5\cdot y:$

$\# \because SA2A3 = -\frac{5}{2}\cdot A1 + 0\cdot A2 + \left(-5\,y + \frac{25}{2}\right)\cdot A3 + 1\cdot A4$

Choose a pair $\{A1, A4\}$**.**

$SA1A4 := expand\left(\frac{x^2}{2\cdot x^2}\cdot(A1) - \frac{x^2}{58\cdot x}\cdot(A4)\right);$

$$\frac{139}{58}\,x\,y - \frac{33}{29}\,x + y^2 + \frac{3}{2}\,y + \frac{3}{2} + \frac{6}{29}\,x\,y^2 - \frac{19}{58}\,x\,y^3 \quad (7)$$

$58\cdot\%;$

$$139\,x\,y - 66\,x + 58\,y^2 + 87\,y + 87 + 12\,x\,y^2 - 19\,x\,y^3 \quad (8)$$

$SA1A4 := -\frac{19}{58}\cdot x\cdot y^3 + \frac{6}{29}\cdot x\cdot y^2 + \frac{139}{58}\cdot x\cdot y - \frac{33}{29}\cdot x + y^2 + \frac{3}{2}\cdot y + \frac{3}{2}:$

$\# \because SA1A4 = 0\cdot A1 + 0\cdot A2 + \left(\frac{19}{58}\cdot y^2 - \frac{107}{58}\cdot y + \frac{396}{58}\right)\cdot A3 + 33\cdot A4 + \left(-\frac{4}{58}\right)\cdot\left(19\cdot y^4 + 83\cdot y^3 + 62\cdot y^2 - 49\cdot y + 69\right)$

$\# \therefore$ $\mathbf{A5 := 19\cdot y^4 + 83\cdot y^3 + 62\cdot y^2 - 49\cdot y + 69}$

$S := \overline{S(A2, A4)}^{\,G'} = 0$

$S := \overline{S(A3, A4)}^{\,G'} = 0$

$S := \overline{S(A1, A5)}^{\,G'} = 0$

$S := \overline{S(A2, A5)}^{\,G'} = 0$

$S := \overline{S(A3, A5)}^{\,G'} = 0$

$S := \overline{S(A4, A5)}^{\,G'} = 0$

Hence $G = \{A1, A2, A3, A4, A5\}$ is Groebner basis.
Note that $G = \{A4, A5\}$ is the "minimal" Groebner basis.