A Case Study of Elliptic Functions in a CAS: Jeffery-Hamel Flow with Maple

Robert M. Corless*
Ontario Research Centre for Computer Algebra
and the Department of Applied Mathematics
University of Western Ontario
London, Ontario, N6A 5B7, Canada.
rcorless@uwo.ca

Dawit Assefa
Department of Applied Mathematics University
of Western Ontario
London, Ontario, N6A 5B7, Canada.
dhaile@uwo.ca

ABSTRACT

Elliptic functions and integrals are not as widely known in the scientific populace as they have been in the past, even though they can still be used to solve an impressive array of applied problems. Computer Algebra Systems (CAS) have stepped into this knowledge gap and claim to provide access to many of the formulae that have proven so useful. This paper takes a classical problem in two-dimensional fluid flow—namely, flow into or out of a wedge-shaped channel with a sink or source at the vertex, which flow is known as Jeffery-Hamel flow and has 'well-known' solutions containing elliptic functions—and tries to duplicate, or even extend, the classical solutions by using a CAS, in this instance Maple. The purposes of this case study include examining just how good CAS can be at elliptic functions; and, more importantly, identifying needs for improvement. Another purpose is to compare the analytical solution with modern numerical solutions. Finally, we believe that this work will motivate improvements to CAS facilities for automatic case analysis. As an aside, we present some simple methods for integration of elliptic functions that seem not to be widely known.
Key words: Elliptic function, Jeffery-Hamel flow, analytic solution, automatic analysis

1. INTRODUCTION

In the Ph.D. thesis [11] we find Jeffery-Hamel flow used as asymptotic boundary conditions to examine steady two-dimensional flow of a viscous fluid in a channel. Jeffery-Hamel is an exact similarity solution of the Navier-Stokes equations, in the special case of two-dimensional flow through a channel with inclined plane walls meeting at a vertex and with a source or sink at the vertex. A wealth of information and references about Jeffery-Hamel flow can be found in [2], where, for example, we find references to [4] and [10], both of which use elliptic functions to study Jeffery-Hamel flow.

*Corresponding Author

ISSAC 2007 Waterloo, CANADA

We confine ourselves here to certain symmetric solutions of the flow, although asymmetric solutions are both possible and of physical interest [12]. Our main aim is to study how handy a CAS is with elliptic functions, not to study Jeffery-Hamel flow, *per se*.

Consider the following quote from [11, p. 15]: "A general solution of these equations can be obtained in terms of elliptic functions." A numerical solution of the equations is used in the thesis [11], but we will see how far we can take the analytical solution using Maple.

2. FIRST ATTEMPT

After nondimensionalization of the form given in [11], the third order ordinary differential equation defining the similarity solutions of Jeffery-Hamel flow in a (two-dimensional) wedge-shaped channel is

$$f''' + 2ff' + 4f' = 0\ , \tag{1}$$

and the boundary conditions are

$$f'(0) = 0\ , \quad f(\pi/2) = 0\ , \quad \text{and} \quad \int_0^{\pi/2} f(\theta)d\theta = -\frac{2}{3}\mathrm{Re} \tag{2}$$

where Re is the Reynolds number.[1]

Of course, we would be most happy with a purely "automatic" solution, that proceeded without human intervention: but at present, the natural first attempt to solve equation (1), by simply calling Maple's `dsolve` command, just gives something like the following implicit formulation as "the answer":

$$t = \int_0^{F(t)} \frac{\pm 3\, dy}{\sqrt{18C_1 + 18c\,y - 36y^2 - 6\mathrm{Re}\,y^3}} - C_2\ .$$

From here, applying the boundary conditions (2) seems quite difficult.

This result does show that the quote from [11] is correct, for this is indeed an elliptic integral. Nonetheless the result is a bit disappointing, because Maple *does* know about elliptic functions and we might have expected this integral

[1]In contrast, the nondimensional formulation in [2] is

$$f''' + 2\alpha R f f' + 4\alpha^2 f' = 0\ ,$$

with the boundary conditions $f = 0$ at $t = \pm 1$, and (which is unusual) $f' = 0$ at $f = 1$. These formulations can be shown to be equivalent.

to evaluate; but the difficulty is that the polynomial contains symbols, and one needs to *factor the cubic* before real progress can be made.

3. DEDUCTIONS

After thought and experimentation, we essentially find ourselves retracing the steps in [2], because it helps the human analysis to do some of the algebra ourselves.

We begin by integrating (1) once to get

$$f'' + f^2 + 4f = z, \quad \text{constant} \tag{3}$$

By using $f(\pi/2) = 0$ we may identify z as $f''(\pi/2)$ but this is itself unknown as yet. We will use z as one of our two primary parameters.

Multiplying (3) by f' and integrating again we get

$$\frac{1}{2}(f')^2 + \frac{1}{3}f^3 + 2f^2 - zf = \kappa, \quad \text{constant.} \tag{4}$$

Now we may begin our analysis. Using $f(\pi/2) = 0$ again, we see that $\frac{1}{2}(f'(\pi/2))^2 = \kappa \geq 0$, which we will use later, and indeed Batchelor uses this to help derive some qualitative features of the flow. At $\theta = 0$, $f' = 0$ and so (putting f_0 for $f(0)$) we have

$$\frac{1}{3}f_0^3 + 2f_0^2 - zf_0 = \kappa \geq 0\,. \tag{5}$$

Therefore, using this in (4), on eliminating κ we get

$$\frac{1}{2}(f')^2 = -\frac{1}{3}(f-f_0)(f^2+ff_0+f_0^2) - 2(f-f_0)(f+f_0) + z(f+f_0)$$

or $(f')^2 = p(f)$ where

$$p(f) := -\frac{2}{3}(f - f_0)\left(f^2 + (f_0+6)f + f_0^2 + 6f_0 - 3z\right)\,. \tag{6}$$

Notice that we have explicitly parameterized one zero of the polynomial $p(f)$, namely f_0. We will see that the two parameters, z and f_0, will uniquely specify the solution, and that we will wish to choose them in order to satisfy the two remaining boundary conditions.

The discriminant of the quadratic factor is

$$\begin{aligned} \Delta &= (f_0+6)^2 - 4(f_0^2 + 6f_0 - 3z) \\ &= 36 - 12f_0 - 3f_0^2 + 12z \end{aligned} \tag{7}$$

We will only investigate cases for which $\Delta > 0$ in this paper, though we first look briefly at the case $\Delta = 0$ below.

Batchelor distinguishes two physical cases, flow in a converging channel and flow in a diverging channel; it turns out that in some cases the solutions of the equations predict both 'inflow' and 'outflow', that is, $f(\theta) < 0$ is possible, as is $f(\theta) > 0$, both in the same flow. Batchelor then deduces that while positive and negative f are possible, it is not possible to have adjacent extrema $f'(\theta) = 0$ with the same sign of f. We shall not need that conclusion, but note that physically reasonable answers to the problem may have both negative and positive f, with possibly many local maxima in the flow.

3.1 Multiple Roots

Let us first consider the case of *multiple roots*: In the first instance, let us consider $f_2 = f_1$, which happens when $\Delta = 0$ and so

$$z = -3 + f_0 + f_0^2/4 = (f_0+6)(f_0+2)/4$$

which gives

$$f_2 = f_1 = -3 - f_0/2\,.$$

If $-6 < f_0 < -2$, then we have $f_0 < -2 < f_2 = f_1 < 0$, but if $f_0 > -2$, we have $f_2 = f_1 < -2 < f_0$, and if $f_0 < -6$ we have $f_0 < -6 < 0 < f_1 = f_2$. Of course we have a triple root if $f_0 = -2$. In the first case, $-6 < f_0 < -2$, integration of (8) gives a solution with $f(0) = f_0$, $f'(0) = 0$, but for which there is no value of f_0 in $(-6, -2)$ that allows $f(\pi/2) = 0$. In detail: integration of $1/\sqrt{p}$ from f_0 (where $\theta = 0$) to f gives a complex arctan,

$$\theta = -6\,i \arctan\left(\frac{\sqrt{6\,f(\theta) - 6\,f_0}}{\sqrt{6\,f_0 - 6\,f_2}}\right)\frac{1}{\sqrt{6\,f_0 - 6\,f_2}}$$

which can be inverted to give (with $f_2 = f_1 = -3 - f_0/2 < f_0 < -2$),

$$\begin{aligned} f(\theta) \;=\; & \left(1 + 3/2\left(\tan\left(1/6\,\theta\,\sqrt{-3/2\,f_0 - 3}\sqrt{6}\right)\right)^2\right) f_0 \\ & +3\left(\tan\left(1/6\,\theta\,\sqrt{-3/2\,f_0 - 3}\sqrt{6}\right)\right)^2 \end{aligned}$$

which when evaluated at $\theta = \pi/2$ and plotted on $-6 < f_0 < -2$ is never zero (but is singular at $f_0 = -6$).

Similarly, when $f_0 > -2$, we get a form containing hyperbolic trig functions for $f(\theta)$, which has $f(0) = f_0$ and $f'(0) = 0$ but for no value of $f_0 > -2$ can be made to have $f(\pi/2) = 0$. If instead $f_0 < -6$, we get

$$\begin{aligned} f(\pi/2) \;=\; & \left(1 + 3/2\left(\tan\left(1/12\,\pi\,\sqrt{-3/2\,f_0 - 3}\sqrt{6}\right)\right)^2\right) f_0 \\ & +3\left(\tan\left(1/12\,\pi\,\sqrt{-3/2\,f_0 - 3}\sqrt{6}\right)\right)^2 \end{aligned}$$

which again by plotting we see is never zero for any $f_0 < -6$.

For the remaining multiple root case, $f_0 = f_1 = f_2 = -2$, the solution of the nonlinear equation is simply $f(\theta) = -2$, which again cannot be made to be zero at $\theta = \pi/2$.

3.1.1 *The cases* $f_2 = f_0$ *or* $f_0 = f_1$

In the case $f_0 = f_2$, we have

$$f'(\theta) = \pm(f(\theta) - f_0)\sqrt{\frac{2}{3}(f_1 - f(\theta))}$$

and the initial condition $f(0) = f_0$ guarantees that the solution will always be $f(\theta) = f_0$, constant. Thus we will be unable to have $f(\pi/2) = 0$ unless $f_0 = 0$. Similarly in the case $f_0 = f_1$. We regard this as a singular limit.

3.2 Simple Roots

Let us next consider the more interesting case $f_0 < 0$ and no extrema in $0 < \theta < \pi/2$. We will see that we now we have enough information to prove that the discriminant (7) is positive and thus all roots are real. From (4) we have

$$f_0\left(\frac{1}{3}f_0^2 + 2f_0 - z\right) \geq 0$$

and since $f_0 < 0$ we have

$$\frac{1}{3}f_0^2 + 2f_0 - z \leq 0$$

or $z \geq \frac{1}{3}f_0^2 + 2f_0$, say $z = \frac{1}{3}f_0^2 + 2f_0 + \frac{1}{12}\varepsilon^2$.

Thus in the discriminant (7) we see

$$\begin{aligned}\Delta &= 36 - 12f_0 - 3f_0^2 + 12z \\ &= 36 - 12f_0 - 3f_0^2 + 4f_0^2 + 24f_0 + \varepsilon^2 \\ &= 36 + 12f_0 + f_0^2 + \varepsilon^2 \\ &= (6 + f_0)^2 + \varepsilon^2 \geq 0\,,\end{aligned}$$

as claimed.

Therefore, in this case, all roots are real. Put $f_2 = \frac{-(f_0+6)-\sqrt{\Delta}}{2}$ and $f_1 = \frac{-(f_0+6)+\sqrt{\Delta}}{2}$. The first order equation for $f(\theta)$ now becomes

$$\left(f'(\theta)\right)^2 = -\frac{2}{3}(f - f_0)(f - f_1)(f - f_2) \tag{8}$$

We will now order these real roots.

THEOREM 1. *If there are no extrema in the interior of the flow field, then* $f_2 \leq f_0 \leq f \leq 0 \leq f_1$.

Proof: Clearly $f_2 \leq f_1$, with equality only if $\Delta = 0$. Since there are no extrema in the interior by hypothesis, we know that f increases from f_0 to 0 by the boundary conditions; hence $f_0 \leq f \leq 0$. For the others, recall that $\Delta = (f_0 + 6)^2 + \varepsilon^2$ and hence $\sqrt{\Delta} \geq |f_0 + 6|$. Therefore $f_2 \leq 0$ and $f_1 \geq 0$. Further, since no extremum occurs in $f_0 < f < 0$, and $f' = 0$ when $f = f_2$, we must have $f_2 \leq f_0$. ∎

We now have enough information to give a correct and useful solution to (8) in terms of elliptic integrals.

4. SOLUTION

If we do not tell Maple the ordering $f_2 < f_0 < 0 < f_1$, then Maple gives the following solution to (8), (ψ is an integration constant): $\theta + \psi = \pm\int \frac{df}{\sqrt{p(f)}} =$

$$\frac{\pm 6(f_1 - f_0)\sqrt{\frac{f-f_0}{f_1-f_0}}\sqrt{\frac{f-f_2}{f_0-f_2}}\sqrt{\frac{f-f_1}{f_0-f_1}}\,F\left(\sqrt{\frac{f-f_0}{f_1-f_0}}, \sqrt{\frac{f_0-f_1}{f_0-f_2}}\right)}{\sqrt{-6(f-f_0)(f-f_1)(f-f_2)}} \tag{9}$$

But this has the problem that the elliptic modulus $k = \sqrt{\frac{f_0-f_1}{f_0-f_2}}$ is purely imaginary since $f_2 \leq f_0 \leq f \leq 0 \leq f_1$. One could then use [1, eq. 17.4:17] to rewrite this by hand as $\theta + \psi =$

$$\frac{-\sqrt{6}}{\sqrt{f_1 - f_2}}F\left(\sqrt{\frac{f_1 - f}{f_1 - f_0}}\,, \sqrt{\frac{f_1 - f_0}{f_1 - f_2}}\right) \tag{10}$$

(resolving sign ambiguities as we will see, and giving modulus $k = \sqrt{\frac{f_1-f_0}{f_1-f_2}}$ with $0 \leq k \leq 1$ as desired).

Rather than recapitulate this hand transformation, necessary with earlier versions of Maple, one could simply verify that with this formula $\frac{d\theta}{df} = \frac{1}{\sqrt{p(f)}}$; since we have already established that $\frac{df}{d\theta} \geq 0$ for our solution, we see that the sign is correct.

Differentiating equation (10) we have

$$\begin{aligned}\frac{d\theta}{df} &= \sqrt{3/2}\frac{1}{\sqrt{f-f_0}\sqrt{f-f_2}\sqrt{f_1-f}} \\ &= \frac{1}{\sqrt{-\frac{2}{3}(f-f_0)(f-f_2)(f-f_1)}} = \frac{1}{\sqrt{p(f)}}\end{aligned}$$

which shows that this formula gives the solution, as desired.

We may now impose the boundary condition $f(\pi/2) = 0$ to get

$$\frac{\pi}{2} + \psi = \frac{-\sqrt{6}}{\sqrt{f_1 - f_2}}F\left(\sqrt{\frac{f_1}{f_1 - f_0}}, \sqrt{\frac{f_1 - f_0}{f_1 - f_2}}\right) \tag{11}$$

and since $\theta = 0$ at $f = f_0$, equation (10) gives

$$\psi = \frac{-\sqrt{6}}{\sqrt{f_1 - f_2}}F\left(\sqrt{\frac{f_1 - f_0}{f_1 - f_0}}, k\right) = \frac{-\sqrt{6}}{\sqrt{f_1 - f_2}}K(k) \tag{12}$$

where K is the complete elliptic integral of the first kind.

We may solve this for f:

$$\sqrt{\frac{f_1 - f}{f_1 - f_0}} = \mathrm{sn}\left(\frac{-\sqrt{f_1 - f_2}}{\sqrt{6}}(\theta + \psi)\right)$$

or

$$f = f_1 - (f_1 - f_0)\mathrm{sn}^2\left(K(k) - \frac{\theta\sqrt{f_1 - f_2}}{\sqrt{6}}, k\right) \tag{13}$$

where $k = \sqrt{\frac{f_1-f_0}{f_1-f_2}}$ satisfies $0 \leq k \leq 1$ and sn is a Jacobian elliptic function. An alternative expression for the boundary condition $f(\pi/2) = 0$ is, therefore, putting $\theta = \pi/2$,

$$f_1 - (f_1 - f_0)\mathrm{sn}^2\left(K(k) - \frac{\pi\sqrt{f_1 - f_2}}{2\sqrt{6}}, k\right) = 0 \tag{14}$$

Note that f_1, k, and f_2 are all known in terms of z and f_0.

4.1 Solution in Maple

In Maple 11, this hand analysis can be carried out by using the `assume` facility, as follows: with $P = -2/3(f - f_0)\cdot(f - f_1)\cdot(f - f_2) = (f')^2$, separation of variables gives the following integral:

```
> int( 1/sqrt(P), f=f0..f )
assuming f2 < f0, f0 < f, f < 0, 0 < f1;
```

which results in something whose presentation can be cleaned up to be: $\theta + \phi =$

$$\sqrt{\frac{6}{(f_1 - f_2)}}\,F\left(\sqrt{\frac{(f_1 - f_2)\,(f - f_0)}{(f_1 - f_0)\,(f - f_2)}}, \sqrt{\frac{f_1 - f_0}{f_1 - f_2}}\right) \tag{15}$$

where here F is the elliptic integral of the first kind. At $f = f_0$, $\theta = 0$, so $\phi = 0$.

Now, in Maple, the elliptic integral of the first kind is defined as $F(x, k) = \int_0^x \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$ using the modulus k. More information on the elliptic function of the first kind $F(x, k)$ appearing in the above equation can be found in the book [8], where it is denoted $F(\phi, k)$. That book also contains a wealth of information on the Jacobian elliptic function $\mathrm{sn}(u, k)$ which appears in our calculations later. *Nota Bene*: Alternative notations in [1] and elsewhere mean that human care is needed in interpretation: the conventions

$$F(\phi/m) = F(\phi, m) = \int_0^\phi \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}$$

with $m = k^2$ (parameter) and $x = \sin\phi$ and

$$F(\phi/\alpha) = \int_0^\phi \frac{d\theta}{\sqrt{1 - \sin^2\alpha\sin^2\theta}}$$

with $k = \sin\alpha$, $\alpha =$ “modular angle”, *overload* the elliptic integral symbol F: it takes on different meanings depending on the alphabet of its argument symbols, surely a confusing state of affairs. We use the $F(x, k)$ notation.

In Maple, the solving step, finding an expression for f in terms of θ from equation (15), is possible but the answer one gets is somewhat different in form (and longer) from equation (13). After some manual simplifications, from (15) we get

$$f = \frac{f_0 - f_2 k^2 \mathrm{sn}^2(\theta\sqrt{f_1 - f_2}/\sqrt{6}, k)}{1 - k^2 \mathrm{sn}^2(\theta\sqrt{f_1 - f_2}/\sqrt{6}, k)} . \tag{16}$$

Using various identities, one can further transform this by hand to become equation (13), but forcing Maple to do it is quite arduous. There is reason to do so, because Maple cannot symbolically integrate the above form, whereas it can integrate equation (13). This certainly represents a weakness in the implementation of the Risch algorithm for integrating trig functions with algebraic extensions (see section (4.2 below), and possibly a bug.

4.2 Integration of Jacobian Elliptic Functions

The following technique was presented by RMC to a joint lab meeting of ORCCA in 2001, but has not, so far as we know, been published elsewhere, and is thus included here. The change of variable shown converts elliptic functions to elementary trigonometric functions. This provides an alternative to the differential algebra techniques of [7]. We begin with an observation: If $f(u) = \mathrm{sn}(u, k)$, then $f(u) = \sin(\mathrm{am}(u, k))$ and similarly $\mathrm{cn}(u, k) = \cos(\mathrm{am}(u, k))$; moreover,

$$\frac{dv}{du} = \mathrm{dn}(u, k)$$

if $v = \mathrm{am}(u, k)$. We will also need the following identities [8]:

$$\begin{aligned} \frac{d}{du}\mathrm{dn}(u, k) &= -k^2\mathrm{sn}(u, k)\mathrm{cn}(u, k) && (17)\\ \mathrm{sn}^2(u, k) + \mathrm{cn}^2(u, k) &= 1 && (18)\\ \mathrm{dn}^2(u, k) - k^2\mathrm{cn}^2(u, k) &= k'^2 = 1 - k^2 && (19)\\ \frac{d}{du}E(u, k) &= \mathrm{dn}^2(u, k) . && (20) \end{aligned}$$

Therefore, when presented with any integral of the form $\int F(\mathrm{sn}(u, k), \mathrm{cn}(u, k), \mathrm{dn}(u, k))du$ one may put $v = \mathrm{am}(u, k)$, whence $dv/\mathrm{dn}(u, k) = du$, and use equation (19) for example to remove the dn, at the cost of a square root algebraic extension; and then any remaining Jacobian elliptic functions become trigonometric functions. After this, integration may proceed by any method.

Remark. As with many computer algebra integration algorithms, this one (and the one of [7]) may sometimes produce spuriously discontinuous integrals, for example $\int \mathrm{dn}(u, k)/(2+\mathrm{sn}(u, k))\, du$ which reduces to an integral that only Derive gets right. See [5] for a substantive discussion of continuous antidifferentiation.

As an example, consider the computation of $\int \mathrm{sn}(u, k)\, du$. After changing the variable as discussed, this becomes

$$\int \frac{\sin(v)}{\sqrt{1 - k^2 + k^2\cos^2(v)}}\, dv$$

and this can (in theory) be evaluated using the standard Risch algorithm. In Maple, this gives

$$-\frac{1}{k}\ln\left(k\cos(v) + \sqrt{1 - k^2 + k^2\cos^2(v)}\right)$$

which can be translated back into the elliptic function form. Even better for this example, the techniques of [7] are already implemented in Maple 11, and can in this case succeed in directly giving the desired answer, $\ln(\mathrm{dn}(u, k) - k\mathrm{cn}(u, k))/k$.

4.3 Imposing the final boundary condition

Either with the Maple solution or the hand solution we may now impose the final boundary condition $\int_0^{\pi/2} f = -\frac{2}{3}\mathrm{Re}$ to get

$$-\frac{2}{3}\mathrm{Re} = \int_0^{\pi/2} f_1 - (f_1 - f_0)\mathrm{sn}^2\left(K(k) - \frac{\theta\sqrt{f_1 - f_2}}{\sqrt{6}}, k\right) d\theta$$

But $\int \mathrm{sn}^2(u, k)du$ is known, either by [1, eq. 16.26.1] or by conversion to an elementary integrand via the change of variable $v = \mathrm{am}(u, k)$ or by the direct techniques of [7].

We have, after using either algorithm discussed above,

$$\int \mathrm{sn}^2(u, k)du = (u - E(\mathrm{sn}(u, k), k))/k^2$$

where $E(x, k) = \int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}dt$ is the elliptic integral of the second kind (with the same notational obfuscations with ϕ, m and α as before). Thus putting $u = K(k) - \theta\frac{\sqrt{f_1 - f_2}}{\sqrt{6}}$ so $d\theta = \frac{-\sqrt{6}}{\sqrt{f_1 - f_2}}du$ we arrive at

$$\begin{aligned} -\frac{2}{3}\mathrm{Re} &= f_1\frac{\pi}{2} + (f_1 - f_0)\frac{\sqrt{6}}{\sqrt{f_1 - f_2}}\left[\frac{u - E(\mathrm{sn}(u, k), k)}{\frac{f_1 - f_0}{f_1 - f_2}}\right]_{\theta=0}^{\theta=\frac{\pi}{2}}\\ &= f_1\frac{\pi}{2} + \sqrt{6}\sqrt{f_1 - f_2}\left[-\frac{\pi}{2}\frac{\sqrt{f_1 - f_2}}{\sqrt{6}}\right]\\ &\quad +\sqrt{6}\sqrt{f_1 - f_2}\Bigg[E(\mathrm{sn}(K(k), k), k)\\ &\quad -E\left(\mathrm{sn}(K(k) - \frac{\pi}{2}\frac{\sqrt{f_1 - f_2}}{\sqrt{6}}, k), k\right)\Bigg]\\ &= f_2\frac{\pi}{2} + \sqrt{6}\sqrt{f_1 - f_2}\,[E(k) - E(\mathrm{sn}(\tilde{u}, k), k)] \end{aligned} \tag{21}$$

where $\tilde{u} = K(k) - \frac{\pi}{2}\frac{\sqrt{f_1 - f_2}}{\sqrt{6}}$. This may be simplified further, in some regions, in that $E(\mathrm{sn}(\tilde{u}, k), k)$ is equal to $E\left(\sqrt{\frac{f_1}{f_1 - f_0}}, k\right)$ over a large region in the (u, f_0, z) space, modulo equation (14); if true for our particular case, this gives

$$-\frac{2}{3}\mathrm{Re} = f_2\frac{\pi}{2} + \sqrt{6}\sqrt{f_1 - f_2}\left[E(k) - E\left(\sqrt{\frac{f_1}{f_1 - f_0}}\,.k\right)\right] \tag{22}$$

All quantities appearing in square roots are nonnegative in this paper. Equation (22) may be recast as

$$\frac{\left(-\frac{2}{3}\mathrm{Re} - \frac{\pi}{2}f_2\right)}{\sqrt{6}\sqrt{f_1 - f_2}} = E(k) - E\left(\sqrt{\frac{f_1}{f_1 - f_0}}, k\right)$$

or

$$E\left(\sqrt{\frac{f_1}{f_1 - f_0}}, k\right) = E(k) + \frac{\left(\frac{2}{3}\mathrm{Re} + \frac{\pi}{2}f_2\right)}{\sqrt{6}\sqrt{f_1 - f_2}}$$

and then if we could invert E as we did F we could get yet another alternative formulation.

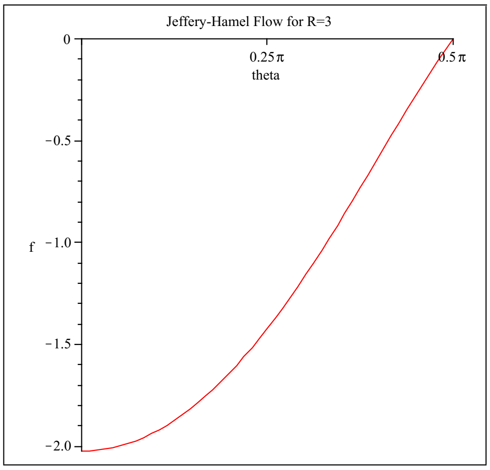

Figure 1: Jeffery-Hamel flow for $\mathrm{Re} = 3$. Negative flow velocity means flow into a converging channel.

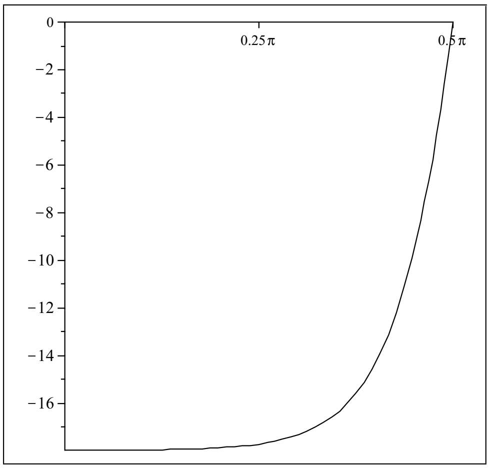

Figure 2: Jeffery-Hamel flow for $f_0 = -17.958$, $z = 250.85$, giving $\mathrm{Re} = 37$. As before, we see pure inflow, with a very flat profile across most of the channel.

4.4 Solving the transcendental equations

Now we have two transcendental equations, (14) and (22), or equivalent equations (11) and (21), to solve for f_0 and $z = f''(\pi/2)$, such that $f(\pi/2) = 0$ and $\int_0^{\pi/2} f(\theta)d\theta = -\frac{2}{3}\mathrm{Re}$. Given a numerical value for Re, we may do so numerically. This may be done in Maple without further human work by use of `fsolve`; clearly for $\mathrm{Re} = 0$ the solutions are $f_0 = z = 0$ and this may be used as a starting guess for small Re; by continuation we may increase Re. *It turns out that there are sometimes multiple solutions.*

An alternative way to proceed is to notice that equation (21) is linear in Re, and so if we suppose that f_0 (or alternatively z) is given, then equation (14) becomes a single transcendental equation to solve for z (alternatively f_0), and then equation (21) defines the Reynolds number of the flow for which this applies. This can be used to construct a table of such flows, or to reduce the solution of the bivariate system to solving a sequence of univariate systems.

Remark. For a given z (or f_0), the equation (14) sometimes has several solutions for f_0 (or z). These give different Re from the remaining equation, and indeed different flow characteristics. We will see an example where equations (21) and (22) give the same Re and flow, but we will also see examples where they differ.

For example, if we ask `fsolve` to find z and f_0 such that both equations (14–21) or (14 and 22) are satisfied for $\mathrm{Re} = 3$, by commands similar to

```
> fsolve( eval( {e1,e2}, R=3), {z, f0} );
```

we find relatively quickly that both pairs give $f_0 = -2.0249$ and $z = -2.0260$ work (printing only a few places). The solution becomes

$$f(\theta) = 0.44589 - 2.4708\,\mathrm{sn}^2\left(-1.8606 + 0.90063\,\theta, 0.71252\right)$$

which matches the desired boundary conditions and shows a monotone inflow. See Figure 1.

We may then use this solution as an initial guess for, say, $\mathrm{Re} = 4$:

```
> fsolve( eval({e1,e2},R=4), {z=-2.026, f0=-2.025});
```

and then use the result of this as an initial guess to find the solution for $\mathrm{Re} = 5$, and so on. This process is known as *simple continuation*. Taking similar steps up to $\mathrm{Re} = 37$, we find $f0 = -17.958$ and $z = 250.85$ give a similar monotone flow. See Figure 2.

By comparison, asking `dsolve` to solve equation (24) numerically, using Allan Wittkopf's sophisticated automatic continuation code[2], we find good agreement. In a form suitable for `dsolve`, the equation is

$$\frac{d^4}{dt^4}F(t) + 2\left(\frac{d}{dt}F(t)\right)\frac{d^2}{dt^2}F(t) + 4\,\frac{d^2}{dt^2}F(t) = 0 \qquad (23)$$

and is subject to the boundary conditions

$$\left\{F''(0) = 0, F'(\pi/2) = 0, F(0) = 0, F(\pi/2) = -2\mathrm{Re}/3\right\}, \qquad (24)$$

then we have $f(\theta) = F'(\theta)$. Numerical integration proceeds quickly (more quickly than the solution of the two equations for z and f_0), and we get a numerical solution that agrees with the analytical solution to better than one part in 10^8. See Figure (3).

Moreover, the numerical method has very little trouble with larger Re, producing a solution for $\mathrm{Re} = 1000$ in not much more computer time than it did for $\mathrm{Re} = 3$; whereas the numerical solution of the transcendental equations for z and f_0 soon runs into trouble, and it is difficult to continue the solution past $\mathrm{Re} = 20$ even with good initial guesses.

In fact, the difficulty is more than merely numerical. It turns out that the analytic formula produced above for equation (21) appears to have a branch cut problem: for large enough Re, spurious solutions are introduced. This is an artifact of the integration method used, of course, but at present, computer algebra systems do not generally produce continuous antiderivatives, or even integrals continuous in a parameter.

[2]This code is available in Maple already since Maple 10, and apparently uses an interesting geometric method to construct an initial guess for continuation (homotopy), and uses a sophisticated continuation scheme with step-doubling (Greg Reid, personal communication). To our knowledge, no description of the algorithm has been published.

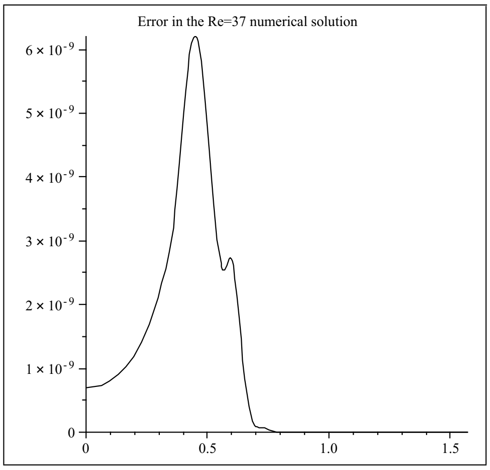

Figure 3: Difference between the numerical solution of equation (24) from the built-in bvp solver in Maple (by Allan Wittkopf), and the 'exact' solution for $Re = 37$.

Table 1: Breakdown of analytic integral formulae (21) and (22)

z	Re (21)	Re (22)	$-3/2 \cdot$ flow integral
320.144	41.557	41.557	41.557
330.41	42.50	-1.0207	-1.0207
641.16	70.87	12.75	-6.391
2891.4	186.87	186.87	-66.647

For example, for $Re = 26.356$, we have a root $f_0 = -10.5178$ and $z = 100$, but the integral of the resulting solution

$$18.308 - 28.826\,\mathrm{sn}^2\left(-2.5690 + 2.3130\,\theta, 0.94765\right)$$

is approximately 4.95, not $-2/3$ of $Re = 26.356$ as it is supposed to be.

For another example, if we take equation (14) and substitute $f_0 = -20$, we see that there are at least four roots $z \approx 320.1444$, $z \approx 330.41$, $z \approx 641.16$, and $z \approx 2891.41$. See Figure 4, and note that the first zero is hard to locate numerically, because the slope is nearly vertical there. The command

```
> fsolve( eval( e1, f0 = -20 ), z = 320 );
```

which gives an initial guess of $z = 320$ to the numerical scheme, *fails*, where

```
> fsolve( eval( e1, f0 = -20 ), z = 320.145 );
```

succeeds. This guess was arrived at interactively by zooming in on the region surrounding the first zero.

All four of these sets of values give positive Δ and order the roots $f_2 < f_0 < 0 < f_1$. Since both equations (21) and (22) are linear in Re, we may use these z values to get valid flows for different Reynolds numbers. However, equations (21) and (22) produce different Re for the same pairs (f_0, z)! Even worse, sometimes *both* formulas give different answers than the numerical integration of $\int_0^{\pi/2} f$, which is supposed to be $-2/3Re$. See Table (1), and the plots in Figures (5–8).

One must conclude that the *symbolic formulae* generated either by hand or with Maple should carry with them some

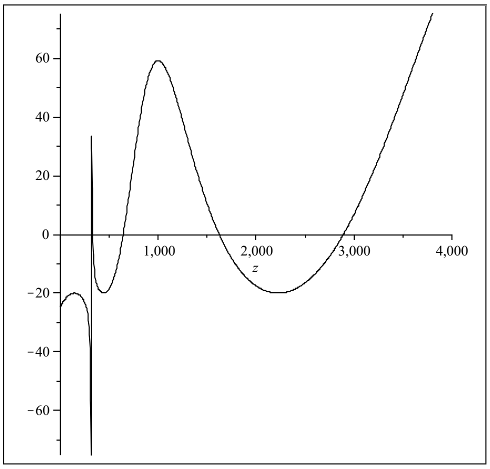

Figure 4: Graph of equation (14) when $f_0 = -20$, for $z > 77$ at which $\Delta = 0$. There are three apparent roots visible, and an apparent singularity. Zooming in on the 'singularity' shows that it is in fact a smooth portion of the curve, giving a zero $z = 320.144$ which has $f_2 = -20.008$, $f_0 = -20$, and $f_1 = 34.008$. Since $f_2 \approx f_0$, we have $k = 0.9999257$, introducing a near-singularity into $K(k)$, because we are getting close to the situation described in Section 3.1.1.

caveats as to when they are valid. This is a serious inconvenience, compared to the simplicity of the numerical solution. Alternative formulations of the equations are possible, but not pursued further here.

5. ASYMPTOTICS

After exploring these solutions, it becomes clear that as Re increases, the flow corresponding to the smallest zero z of equation (14) has a very flat profile, rising only near the boundary $\theta = \pi/2$ to zero. Recall from Section 3.1.1 that the singular solution $f(\theta) = f_0$ corresponds to a double root $f_0 = f_2$. Of course this cannot match the boundary condition $f(\pi/2) = 0$, but this strongly suggests a *singular perturbation* analysis [9]. We note that for the solution plotted in Figure 5 we have $f_2 = -20.008$ and $f_0 = -20$, already very close even just for $Re \approx 42$. So we are tempted to force our solution into that mold: put $z = f_0(f_0 + 4) + \varepsilon$ for $\varepsilon > 0$ (if $\varepsilon = 0$ then this value of z makes $f_2 = f_0$ exactly). Then some simple use of `series` and of `asympt` give the following:

$$f_2 = f_0 + \frac{1}{2 + f_0}\varepsilon + O(\varepsilon^2) \quad (25)$$

$$k = 1 - \frac{1}{6(2 + f_0)}\varepsilon + O(\varepsilon^2) \quad (26)$$

$$K(k) = -\frac{1}{2}\ln\varepsilon + \ln(4\sqrt{3}(-2 - f_0)) + O(\varepsilon\ln\varepsilon) \quad (27)$$

$$E(k) = 1 + O(\varepsilon\ln\varepsilon) \quad (28)$$

$$E(C, k) = C + O(\varepsilon\ln\varepsilon) \quad (29)$$

$$f_0 = -\frac{4}{3\pi}Re + O\left(\sqrt{Re}\right) \quad (30)$$

The last asymptotic conclusion supposes that as $Re \to \infty$ we must have $\varepsilon \to 0$. Essentially, this is the flow due to a singular solution $f(\theta) = f_0$ all the way across the interval,

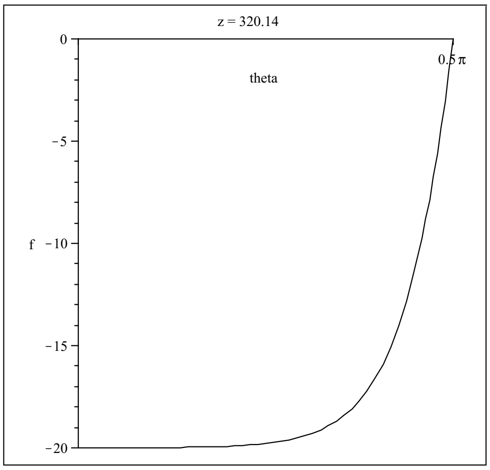

Figure 5: Jeffery-Hamel flow for $f_0 = -20$, $z = 320.144$, giving $\Re = 41.557$. We see pure inflow, though in this case $k = 0.9999257$ and the function $f(\theta)$ contains a singularity if $k = 1$. Note that this is the physically most interesting flow, and it is the most difficult to obtain numerically by solving equation (14), although the direct numerical boundary value method of Allan Wittkopf works very well.

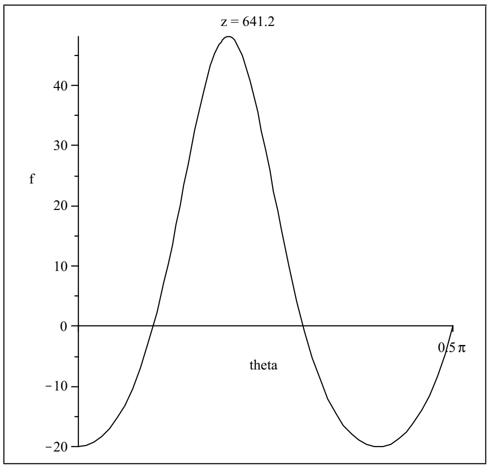

Figure 7: Jeffery-Hamel flow for $f_0 = -20$, $z = 641$. We see both inflow and outflow, which violates one of the assumptions for our formulas. Neither equation (21) nor equation (22) get the correct $\Re$ for this flow, though each is different.

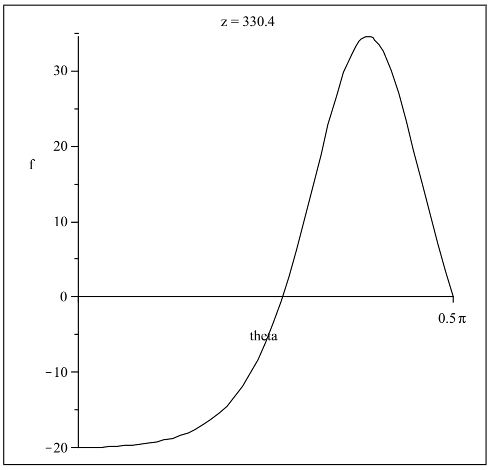

Figure 6: Jeffery-Hamel flow for $f_0 = -20$, $z = 330.41$. We see both inflow and outflow, which violates one of the assumptions for our formula. Equation (22) gets the correct $\Re = -1.02$ for this figure, though equation (21) does not.

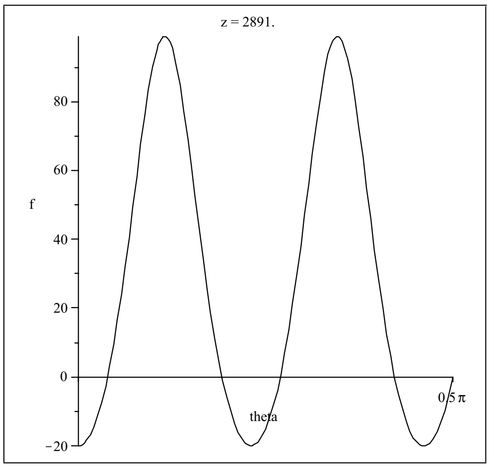

Figure 8: Jeffery-Hamel flow for $f_0 = -20$, $z = 2981$. We see both inflow and outflow, which violates one of the assumptions for our formula. In this case, equations (21) and (22) agree on their prediction for $\Re$; however, both are wrong.

neglecting the boundary layer entirely. Already by $f_0 = -20$ this formula gives $Re = 15\pi \approx 47.123$, which compares reasonably well with the numerical result 41.557 from Table (1), but including the next term gives $Re = 42.18$, which is tolerably accurate. We note that Batchelor came to the same asymptotic estimate by a direct method, bypassing the elliptic functions entirely; here we used series expansion on equation (21) with the choice $z = f_0(f_0 + 4) + \varepsilon$ to find a relation between f_0 and Re; we took it for granted that this solution would also satisfy equation (14) which is quite a bit harder to compute the series expansion of, because of the sn term of a large argument. Nonetheless this represents a useful strength of the symbolic approach.

6. LESSONS LEARNED

- At the moment, automatic solution of Jeffery-Hamel flow is not possible in Maple. Human intervention and analysis are still needed.
- Maple's elliptic integrals are standard: it's just that there are half-a-dozen standards to choose from.
- Symbolic integration of a function containing a Jacobian elliptic function can be carried out algorithmically, but is not yet satisfactory from the point of view of continuous dependence on parameters. This is of course in general an open problem [6].
- The analytical solution of the Jeffery-Hamel flow problem requires numerical solution of two simultaneous transcendental equations. Without further pre-processing, this is more expensive computationally than a direct numerical solution of the equivalent 4th order BVP.
- Some asymptotic analysis can be carried out for large Re, after identification of the proper singular limit. This represents a genuine advantage of a CAS. These results should be very useful in comparing with the asymptotic solutions of these equations presented in [11, Appendix C].
- Why might this (eventually) be uniformly superior to finding a numerical solution to the original boundary value problem? In other words, why are we doing this?
  1. First of all, it might be cheaper to represent the answer: we need only a table of values of z and f_0 for a collection of values of Re. In comparison, the representation of the numerical solution of the differential equation would take more space and be less intelligible.
  2. Some of the more oscillatory solutions, which are dynamically possible flows, seem to be difficult to compute numerically[3]; but they are no harder to compute with this semi-analytic approach than the smooth (flat) profile flow. And indeed if one computes all roots z of the equation, then one gets all possible flows—hence it seems that more information comes from the analytical solution, even if there is a numerical portion of the process.
  3. Once computed, the solution will be cheaper to evaluate at an arbitrary point to high precision (evaluation of the Jacobian elliptic functions is extraordinarily cheap, via the arithmetic-geometric mean, for example [3]).
  4. Finally, superior or not, this process would provide a good check on the numerical answers in [11].

[3] We have not made extensive experiments, but in the limited series that we have tried, the numerical method seems to prefer the flat solution profile.

Acknowledgements

We wish to thank Maciej Floryan for encouraging this research. George Labahn wrote the code in Maple which allows the integration of that elliptic integral in the first place, and put hooks into `solve` so that it could be inverted. Discussion of this problem with David Jeffrey (note his name is spelled differently from the Jeffery in Jeffery-Hamel!) was very helpful. Dhavide Aruliah made helpful comments on an original draft. Allan Wittkopf was kind enough to use this problem as a test problem for his numerical bvp continuation code, and as you can see from the examples here his code works very well. We would also like to thank Pat Malone for typing the manuscript.

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