

Iterated Integration: Define the operator

$$I_a(f)(x) := \int_a^x f(t) dt \quad (1)$$

and the iteration formula

$$I_a^{n+1}(f)(x) := \int_a^x I_a^n(f)(t) dt \quad (2)$$

with the conventions that $I_a^0(f)(x) = f(x)$ and $I_a^1(f)(x) = I_a(f)(x)$. Then we have the following theorem:

Theorem

$$I_a^n(f)(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f(t) dt \quad (3)$$

for $n \in \mathbb{N}$.

Proof: The theorem is true for $n = 1$ by definition. Assume that it is true for $n = k$ for some fixed $k \in \mathbb{N}$. Then by formula (2)

$$I_a^{k+1}(f)(x) = \int_a^x I_a^k(f)(t) dt = \int_a^x \int_a^t \frac{(t-\tau)^{k-1}}{(k-1)!} f(\tau) d\tau dt \quad (4)$$

$$= \int_a^x \int_\tau^x \frac{(t-\tau)^{k-1}}{(k-1)!} f(\tau) dt d\tau \quad (5)$$

$$= \int_a^x f(\tau) \int_\tau^x \frac{(t-\tau)^{k-1}}{(k-1)!} dt d\tau \quad (6)$$

where step (4) follows by the induction hypothesis; in step (5) we change the order of integration, and in step (6) we move $f(\tau)$ outside the integration with respect to t . Since, finally,

$$\int_\tau^x \frac{(t-\tau)^{k-1}}{(k-1)!} dt = \frac{(t-\tau)^k}{k!} \Big|_{t=\tau}^{t=x} = \frac{(x-\tau)^k}{k!} \quad (7)$$

the theorem is proved.

References

- [1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, 1970.
- [2] T. J. Anastasio. The fractional-order dynamics of brainstem vestibulo-oculomotor neurons. *Biological Cybernetics*, 72:69–79, 1994.
- [3] George E. Andrews, Richard Askey, and Ranajan Roy. *Special functions*. 1999.
- [4] Mhenni Benghorbal and Robert M. Corless. The n th derivative. *SIGSAM BULLETIN: Communications on Computer Algebra*, 36(1):10–14, 2002.
- [5] Mhenni Benghorbal and Robert M. Corless. Power series solutions of fractional differential equations. *Intl. J. Maths. Applic.*, 2004.
- [6] Mhenni M. Benghorbal. *Power Series Solutions of Fractional Differential Equations and Symbolic Derivatives*. PhD thesis, The University of Western Ontario, London, 2004.
- [7] G. W. Scott Blair. The role of psychophysics in rheology. *J. of Colloid Sciences*, 2:21–32, 1947.
- [8] David Borwein. A summability factor theorem. *Journal of the London Mathematical Society*, 25:302–315, 1950.
- [9] M. Caputo. Linear model of dissipation whose q is almost frequency independent. *II Geophys. J. R. Astr. Soc.*, 13:529–539, 1967.
- [10] M. Aslam Chaudhry and Syed M. Zubair. *On A Class of Incomplete Gamma Functions with Applications*. Chapman Hall/CRC, 2002.
- [11] Robert M. Corless. Error backward. In Peter Kloeden and Ken Palmer, editors, *Contemporary Mathematics*, volume 172, pages 31–62. American Mathematical Society, 1994.
- [12] Robert M. Corless. *Essential Maple*. Springer-Verlag, New York, 1995.
- [13] Robert M. Corless. *Essential Maple 7*. Springer-Verlag, New York, 2002.

- [14] Robert M. Corless, Gaston H. Gonnet, David E. G. Hare, David J. Jeffrey, and Donald E. Knuth. “On the Lambert W function”. *Advances in Computational Mathematics*, 5:329–359, 1996.
- [15] Robert M. Corless and David J. Jeffrey. Well, it isn’t quite that simple. . . . *SIGSAM Bulletin*, 26(3):2–6, August 1992.
- [16] Matt Davison. *Spatial and Deterministic Limits on Randomness*. PhD thesis, The University of Western Ontario, London, CANADA, 1995.
- [17] Matt Davison and Christopher Essex. Fractional differential equations and initial value problems. *Math. Scientist*, 23:108–116, 1998.
- [18] A. M. A. El-Sayed. Multivalued fractional differential equations. *App. Math and Comput*, 80:1–11, 1994.
- [19] A. M. A. El-Sayed. Fractional order evolution equations. *Journal of Fractional Calculus*, 7:89–100, May 1995.
- [20] Ricardo Estrada and Ram P. Kanwal. *Singular Integral Equations*. Birkhauser, 2000.
- [21] Leonhard Euler. *Opera Omnia*, volume 6 of *Opera Mathematica I*. 1921 (orig. date 1779).
- [22] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete Mathematics*. Addison-Wesley, 1994.
- [23] Godfrey Harold Hardy. *A Course of Pure Mathematics*. Cambridge University Press, 10th edition, 1952.
- [24] Russell Jay Hendel. Differentiation via partial fractions: A case against CAS. *The College Mathematics Journal*, 22(5):415–417, November 1991.
- [25] Peter Henrici. *Applied and Computational Complex Analysis*, volume II. Wiley-Interscience, 1977.
- [26] D.J. Jeffrey, Robert M. Corless, D. E. G. Hare, and Donald E. Knuth. “Sur l’inversion de $y^\alpha e^y$ au moyen des nombres de Stirling associés”. *Comptes Rendus de l’Académie des Sciences, Paris*, 320:1449–1452, 1995.

- [27] D.J. Jeffrey, D. E. G. Hare, and Robert M. Corless. “Unwinding the branches of the Lambert W function”. *Mathematical Scientist*, 21:1–7, 1996.
- [28] Marcia Kleinz and Thomas J. Osler. A child’s garden of fractional derivatives. *The College Mathematics Journal*, 31(2):82–87, 2000.
- [29] Gilbert Labelle. “Sur l’inversion et l’itération continue des séries formelles”. *European Journal of Combinatorics*, 1:113–138, 1980.
- [30] Cornelius Lanczos. *Applied Analysis*. Dover, 1988.
- [31] J. L. Lavoie, T. J. Osler, and R. Tremblay. Fractional derivatives and special functions. *SIAM Review*, 18:240–268, April 1976.
- [32] Xiaorang Li. *Fractional Calculus, Fractal Geometry, and Stochastic Processes*. PhD thesis, The University of Western Ontario, London, CANADA, 2003.
- [33] J. Liouville. Memoire: sur le calcul des differentielles a indices quelconques. *J. de l’Ecole Polytechnique*, 13:71–162, 1832.
- [34] J. Liouville. Memoire: sur quelques questions de géometrie et de mécanique, et sur un nouveau genre pour resoudre ces questions. *J. de l’Ecole Polytechnique*, 13:1–69, 1832.
- [35] A. M. Mathai. *The H-function with Applications in Statistics and Other Disciplines*. John Wiley and Sons, 1978.
- [36] A. M. Mathai. *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*. Oxford Science Publications, 1993.
- [37] Keith B. Oldham and Jerome Spanier. *The Fractional Calculus*. Academic Press, 1974.
- [38] A. Oustaloup. *La Command CRONE*. Hermes, Paris, 1991.
- [39] R. B. Paris and D. Kaminski. *Asymptotics and Mellin-Barnes Integrals*. Cambridge, 2001.
- [40] Igor Podlubny. *Fractional Differential Equations*. Academic Press, San Diego, California, 1999.

- [41] E. L. Post. Generalized differentiation. *Trans. Amer. Math. Soc.*, 32:723–781, 1930.
- [42] Theodore J. Rivlin. *Chebyshev Polynomials*. Wiley-Interscience, 1990.
- [43] Bruno Salvy and Paul Zimmerman. GFUN: A Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software*, 20(2):163–177, June 1994.
- [44] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives, theory and applications*. Gordon and Breach, 1996.
- [45] Michael Unser, Stefan Horbelt, and Thierry Blu. Fractional derivatives, splines and tomography. *preprint*, 2004.
- [46] Z. X. Wang and D. R. Guo. *Special Functions*. World Scientific, 1989.