Algebraic properties of the Lambert W Function (following Liouville and Rosenlicht)

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The Lambert W function [5, 9] is a multi-valued function defined as the solution of

$$W(x)e^{W(x)} = x\,, \tag{1}$$

one of the simplest possible non-algebraic equations. The Wright ω function [4] also satisfies a simple transcendental equation (away from its discontinuities):

$$\omega(x) + \ln \omega(x) = x\,. \tag{2}$$

These are, of course, "implicitly elementary" functions in the sense discussed in [7]. One can ask whether there are explicit formulations of those functions in terms of known functions, or are they genuinely new functions. A common class of "well-known" functions are the Liouvillian functions.

Definition 1 *Let $(k,')$ be a differential field of characteristic* 0*. A differential extension $(K,')$ of k is called* Liouvillian *over k if there are $\theta_1,\ldots,\theta_n \in K$ such that $K = C(x,\theta_1,\ldots,\theta_n)$ and for all i, at least one of the following holds:*

1. *θ_i is algebraic over $k(\theta_1,\ldots,\theta_{i-1})$;*

2. $\theta_i' \in k(\theta_1, \ldots, \theta_{i-1})$*;*

3. $\theta_i'/\theta_i \in k(\theta_1, \ldots, \theta_{i-1})$*.*

We say that $f(x)$ *is a* Liouvillian function *if it lies in some Liouvillian extension of* $(C(x), d/dx)$ *for some constant field* C*.*

It turns out that the possible closed-form expressions for solutions of equations of the form (1–2) were already studied by Liouville [6], who was certainly able to prove already that $W(x)$ is not a Liouvillian function. In any event, this result was known to Rosenlicht, who published in [8] a proposition that can be applied to prove easily that $W(x)$ and $\omega(x)$ (or many functions defined by similar transcendental equations) are not Liouvillian. Yet, questions about whether $W(x)$ is elementary or Liouvillian appear in the literature [3], possibly because Rosenlicht's paper is not as well-read as it deserves to be, so we illustrate in this note how Rosenlicht's theorem can prove that neither $W(x)$ nor $\omega(x)$ are Liouvillian.

We start by recalling Rosenlicht's result.

Proposition 1 *[8, Proposition, p.21] Let* k *be a differential field of characteristic zero and let* $y_1, \ldots, y_n, z_1, \ldots, z_n$ *be elements of a Liouvillian extension of* k *having the same subfield of constants as* k*. Suppose that*

$$\frac{y_i'}{y_i} = z_i', \quad i = 1, \ldots, n\,,$$

and that $k(y_1, \ldots, y_n, z_1, \ldots, z_n)$ *is algebraic over each of its subfields* $k(y_1, \ldots, y_n)$ *and* $k(z_1, \ldots, z_n)$*. Then,* $y_1, \ldots, y_n, z_1, \ldots, z_n$ *are all algebraic over* k*.*

An immediate consequence of the case $n = 1$ of that proposition is that if $W(x)$ and $\omega(x)$ are Liouvillian functions, then they must be algebraic functions: suppose that W belongs to a Liouvillian extension K of $\mathbb{C}(x)$. Take $k = C(x)$ where C is the constant subfield of K, then K is Liouvillian over k and both fields have the same subfield of constants. Taking logarithmic derivatives on both sides of (1) yields

$$W'/W + W' = 1/x\,, \tag{3}$$

whence $y'/y = W'$ where $y = x/W \in K$. Since $k(y, W) = k(y) = k(W)$, Rosenlicht's theorem implies that W is algebraic over $k = C(x)$. The proof is similar for $\omega(x)$: differentiating both sides of (2) yields $\omega' + \omega'/\omega = 1$, whence $\omega'/\omega = z'$ where $z = x - \omega$. Since $k(\omega, z) = k(\omega) = k(z)$, Rosenlicht's theorem implies that ω is algebraic over $k = C(x)$.

There are obvious analytic arguments why $W(x)$ and $\omega(x)$ cannot be algebraic functions, so they cannot be Liouvillian functions: if $W(x)$ has a pole of finite order, then $e^{W(x)}$, and therefore $W(x)e^{W(x)}$, have an essential singularity, so $W(x)e^{W(x)}$ cannot equal x. Similarly if $\omega(x)$ has a zero, then $\ln \omega(x)$, and therefore $\omega(x) + \ln \omega(x)$, have a logarithmic singularity, so $\omega(x) + \ln \omega(x)$ cannot equal x. Since algebraic functions with either no pole or no zero must be constants, and $W(x)$ and $\omega(x)$ cannot be constant, they cannot be algebraic.

The above argument can be cast in algebraic terms. Since Rosenlicht proved his result algebraically, we outline the algebraic proof that $W(x)$ and $\omega(x)$ cannot be algebraic functions. Note that (3) implies that $y = W(x)$ is a solution of the differential equation

$$xy'(1+y) = y\,. \tag{4}$$

We first recall some notations and results from [2]: we say that a field E is an algebraic function field of one variable over a subfield $F \subset E$ if

- E is of transcendence degree 1 over F,
- for any $t \in E$ transcendental over F, $[E : F(t)]$ is finite.

By an F-place of E, we then mean the maximal ideal of a valuation ring of E containing F. For such a place p, we write $\nu_p : E^* \to \mathbb{Z}$ for its order function. It has in particular the following properties:

- $\nu_p(c) = 0$ for any $c \in \overline{F} \cap E^*$.
- $\nu_p(ab) = \nu_p(a) + \nu_p(b)$ and $\nu_p(a+b) \geq \min(\nu_p(a), \nu_p(b))$ for any $a, b \in E^*$.
- $\nu_p(a+b) = \min(\nu_p(a), \nu_p(b))$ for any $a, b \in E^*$ such that $\nu_p(a) \neq \nu_p(b)$.
- For any $a \in E^*$, if $\nu_p(a) \geq 0$ at all the F-places of E, then a is algebraic over F.

Let now $t \in E$ be transcendental over F and p be any F-place of E. We write $r_t(p) \in \mathbb{Z}_{>0}$ for the ramification index of p over $F(t)$. In addition, we call the place p *infinite* (w.r.t. t) if $t^{-1} \in p$, *finite* (w.r.t. t) otherwise.

Proposition 2 *Let $(F, ')$ be a differential field containing an element x such that $x' = 1$. If F has transcendence degree 1 over its constant subfield, then the only solution $y \in F$ of (4) is $y = 0$.*

Proof. Let C be the constant subfield of F and suppose that F has transcendence degree 1 over C. Since $x' = 1$, x is transcendental over C, so F is algebraic over $C(x)$. Let $y \in F$ be a nonzero solution of (4) and $E = \overline{C}(x, y)$, which is an algebraic function field of one variable over $\overline{C}$. Let p be any $\overline{C}$-place of E. Applying ν_p on both sides of (4), we get

$$\nu_p(x) + \nu_p(y') + \nu_p(1+y) = \nu_p(y)\,. \tag{5}$$

Suppose that $\nu_p(y) < 0$. Then, $\nu_p(1+y) = \min(0, \nu_p(y)) = \nu_p(y)$ and (5) becomes

$$\nu_p(x) + \nu_p(y') = 0\,. \tag{6}$$

If p is finite w.r.t. x, then $\nu_p(x) \geq r_x(p)$. But Lemma 1.7 of [1] implies that $\nu_p(y') = \nu_p(y) - r_x(p) < -r_x(p)$, in contradiction with (6). If p is infinite, then $\nu_p(x) = -r_x(p)$. But Lemma 1.8 of [1] implies that $\nu_p(y') \leq \nu_p(y) + r_x(p) < r_x(p)$, in contradiction with (6). Therefore $\nu_p(y) \geq 0$ at all the $\overline{C}$-places of E, which implies that $y \in \overline{C}$, hence that $y' = 0$, and (4) becomes $0 = y$. □

Since the only algebraic solution of (4) is 0, which is not a solution of (1), $W(x)$ cannot be algebraic, hence it cannot be a Liouvillian function.

The proof that $\omega(x)$ is not an algebraic function is similar, since $y = \omega(x)$ is a solution of the differential equation $y'(1 + y) = y$. The equality (5) becomes $\nu_p(y') + \nu_p(1+y) = \nu_p(y)$. If $\nu_p(y) > 0$ for some p, then $\nu_p(1+y) = 0$, so $\nu_p(y') = \nu_p(y)$ and Lemma 1.7 of [1] implies that p must be infinite. Differentiating the Puiseux series $y = \sum_{n \geq \nu_p(y)} c_n x^{-n/r_p}$ shows that $\nu_p(y') > \nu_p(y)$, a contradiction. We conclude as above that the only algebraic solution of $y'(1+y) = y$ is $y = 0$, which implies that $\omega(x)$ cannot be algebraic.

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