Alternative ways of solving polynomial systems

Ilias Kotsireas and Greg Reid
Ontario Research Centre for Computer Algebra
University of Western Ontario
London, Ontario, CANADA N6A 5B7
`Ilias.Kotsireas@orcca.on.ca, Greg.Reid@uwo.ca`
`http://www.orcca.on.ca/`

May 27, 2001

Abstract

We exploit an isomorphic embedding of polynomial systems in linear systems of PDE, to solve polynomials using differential techniques.

In theory such methods should already be present in the polynomial sector of the computer algebra system, and if present should be more efficient, since the overhead of polynomial operations should be lower than that for differential systems.

In practice however, the varying amounts of programming effort in different sectors of a computer algebra systems will *always* mean that some methods will be present in some sectors and not in others. Sometimes this occurs in surprising ways, as in this note.

We suggest that constructing conversion tools between different sectors of a computer algebra system is a valuable activity. We illustrate such tools in this paper to convert polynomial systems to differential systems. Some problems are treated which cannot be solved using the existing polynomial algorithms implemented in the computer algebra system we used (Maple 7).

We present some ways to solve polynomial systems using the differential elimination algorithm `rifsimp` and the notion of normal set which arises in the interpretation of polynomial system solving as a matrix eigenproblem. These ideas apply to zero-dimensional systems, that is to systems with finitely many solutions. For such systems we show how to compute the associated univariate polynomials satisfied by a single variable of the system. We also treat some systems with parameters, for which we identify several cases in parameter space with their associated solutions. These ideas are illustrated with two zero-dimensional polynomial systems that arise in the study of central configurations in the N-body problem of Celestial Mechanics and an inverse kinematics exmaple from Robotics. The computations have been performed in Maple 6.

1 Introduction

To solve a zero-dimensional system an elimination technique is often applied to derive a univariate polynomial satisfied by one of variables. Some examples of such techniques are resultants, triangular sets and Gröbner bases. Another approach to polynomial system solving views the problem as a matrix eigenproblem, see [6] for the original work and [2] for an expository article together with an implementation of a Maple 7 routine that we will use in the sequel. Differential elimination algorithms have been developed by a number of authors and have been applied in many areas. See for example: Boulier et. al [1]; Mansfield [5]; Reid, Wittkopf, Rust and co-workers [7, 8, 9]; Schwarz [10, 12]; Seiler [13]. The software packages of Wittkopf and Boulier are available as part of core functionality of Maple 6 and Maple 7 in the `rifsimp` subpackage of `DEtools`, and in the `diffalg` package. Mansfield's Maple package `diffgrob2` is available from her web site (together with an online demo).

The ideas used in the differential elimination algorithms are similar with the ideas used in classical Gröbner basis theory. In this note we will show how to use matrix eigenproblems and differential elimination to effectively compute univariate polynomials resulting from elimination in zero-dimensional systems.

2 Polynomials and Differential Polynomials

In this section we present a well-known isomorphism which allows us to translate monomials (and thus polynomials, by additivity) to differential monomials and vice versa.

Consider a polynomial in the n variables $x := x_1, \ldots, x_n$ with rational number coefficients. Let $p(x) \in \mathbb{Q}[x_1, x_2, ..., x_n]$ be such a polynomial. The isomorphism embedding such a polynomial into a partial differential expression is:

$$p(x_1, ..., x_n) \overset{\phi}{\longleftrightarrow} p\left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right) u(x_1, ..., x_n) \qquad (1)$$

It is easily verified that the ring properties are preserved under the map ϕ. Multiplication by x_j in the ring $\mathbb{Q}[x_1, ..., x_n]$ is equivalent to applying the differential operator $\frac{\partial}{\partial x_j}$ in the differential ring $\mathbb{Q}\left[\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right]$.

Actually the bijection above can be made more general. Consider $x = (x_1, x_2, ..., x_n)$, and $\partial := \partial_{x_1}, ..., \partial_{x_n}$ then

$$p_1(x)u_1 + ... + p_k(x)u_k \overset{\phi}{\longleftrightarrow} p_1(\partial)u_1(x) + ... + p_k(\partial)u_k(x) \qquad (2)$$

Consequently the mapping extends to one between modules, generated by $u_1, u_2, ..., u_k$.

Simple Example: Consider the simple system of polynomials $xy - 1, x^2y - a$. Then applying the isomorphism ϕ, which we will denote `PolynomialToPDE` in this paper (with inverse `PDEToPolynomial`, is equivalent to $x \mapsto \frac{\partial}{\partial x}, y \mapsto \frac{\partial}{\partial y}$. We obtain the differential system:

$$\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y}\right) u(x, y) - u(x, y), \qquad (3)$$

$$\left(\frac{\partial}{\partial x}\right)^2\left(\frac{\partial}{\partial y}\right) u(x, y) - au(x, y) \qquad (4)$$

To perform elimination on the polynomial system $xy - 1, x^2y - a$, the first equation is multiplied by x and then subtracted from the second to yield $x - a$. Under the isomorphism, this corresponds to differentiating the first PDE with respect to x and subtracting from the second PDE, to obtain the equivalent result $\frac{\partial}{\partial x}u(x, y) - au(x, y)$. Simplification of $xy - 1$ yields $ay - 1$, and the corresponding differential operation yields $a\frac{\partial}{\partial y}u(x, y) - u(x, y)$.

3 Application to the systems $\mathcal{S}_4$ and $\mathcal{S}_5$

The systems $\mathcal{S}_4$ and $\mathcal{S}_5$ below, arise in the study of central configurations in the newtonian planar 4-body problem with equal masses, and the newtonian spatial 5-body problem with equal masses respectively, see [4] for a detailed study.

The system $\mathcal{S}_4$:

$$\boxed{\begin{gathered} -2\,p^3 + 2\,p^3\phi^3 - 4\,\phi^3 sp^2 + 5\,\phi^3 s^3 p - \phi^3 s^5 = 0 \\ -2\,sp^3 - 2\,\phi^3 s^2 + \phi^3 s^4 - 3\,\phi^3 s^2 p + 2\,\phi^3 p = 0 \\ -2\,s^2 + s^4 - 4\,s^2 p + \phi^2 + 1 + 4\,p = 0 \end{gathered}}$$

The system $\mathcal{S}_5$:

$$\boxed{\begin{gathered} -4\,p^3\phi^3 + 6\,p^3 + 12\,p^2 s\phi^3 - 15\,ps^3\phi^3 + 3\,ps\phi^3 - s^3\phi^3 + 3\,s^5\phi^3 = 0 \\ -5\,p\phi^3 + 6\,p^3 s + 9\,ps^2\phi^3 + 5\,s^2\phi^3 - 3\,s^4\phi^3 = 0 \\ 3 + 4\,\phi^2 + 12\,p - 12\,ps^2 - 6\,s^2 + 3\,s^4 = 0 \end{gathered}}$$

The systems $\mathcal{S}_4$ and $\mathcal{S}_5$ are zero-dimensional (i.e. have finitely many solutions). We are interested in computing the associated univariate polynomials in ϕ, because these will give us the possible values for one of the mutual distances between the particles in each case. The computations of the lexicographical Gröbner bases of $\mathcal{S}_4$, $\mathcal{S}_5$ do not terminate in Maple 6. Suppose that the equations of the systems $\mathcal{S}_4$ have been assigned to the Maple variables `S4` and `S5` as lists. The total degree computations can be accomplished either in Maple 7 using `PolynomialToPDE`, `Groebner[SetBasis]` and `Groebner[MulMatrix]` as follows. It turns out that the second approach is sometimes faster.

3.1 Degree computation with the Groebner module in Maple 7

We will use the degree ordering `degOrd := tdeg(s,p,phi)`. This is done simply by typing (see the associated Maple worksheet):

```
 degOrd := tdeg(s,p,phi):
degBasisS4 := Groebner[gbasis](S4, degOrd):
degBasisS5 := Groebner[gbasis](S5, degOrd):
```

3.2 Degree computation with rifsimp in Maple

We do the computations in parallel for both $\mathcal{S}_4$ and $\mathcal{S}_5$.

First, translate the polynomial systems $\mathcal{S}_4$ and $\mathcal{S}_5$ into differential systems using `PolynomialToPDE`:

```
 PolyToPDE := PolynomialTools[PolynomialToPDE]:
desysS4 := PolyToPDE(S4,[s,p,phi],[u]):
desysS5 := PolyToPDE(S5,[s,p,phi],[u]):
```

Then process the differential systems desysS4 and desysS5 with `rifsimp`:

```
 tdegrsysS4 := DEtools[rifsimp](desysS4):
tdegrsysS5 := DEtools[rifsimp](desysS5):
```

Now, map the result back to polynomials using `PDEToPolynomial`:

```
PDEToPoly := PolynomialTools[PDEToPolynomial]:
degBasisS4:= PDEToPoly(tdegrsysS4[Solved],[s,p,phi],[u]):
degBasisS5:= PDEToPoly(tdegrsysS5[Solved],[s,p,phi],[u]):
```

3.3 Compute the univariate polynomials with SetBasis and MulMatrix

Compute the normal sets for both `degBasisS4` and `degBasisS5`.

```
 nsS4, rvS4 := Groebner[SetBasis](degBasisS4,
degOrd):
nsS5, rvS5 := Groebner[SetBasis](degBasisS5,
degOrd):
```

Compute the multiplication matrix for the variable ϕ.

```
 MphiS4 := Groebner[MulMatrix](phi, nsS4, rvS4,
degBasisS4, degOrd):
MphiS5 := Groebner[MulMatrix](phi, nsS5, rvS5,
degBasisS5, degOrd):
```

Compute and factor the characteristic polynomials of the multiplication matrices.

```
 CP := LinearAlgebra[CharacteristicPolynomial]:
p37 := sort(factor( CP(MphiS4,lambda))):
p43 := sort(factor( CP(MphiS5,lambda))):
```

3.4 Systems with parameters

The use of the `rifsimp` algorithm to treat polynomial systems can be exploited further to perform Gröbner bases computations with polynomial systems with parameters. The isomorphism (2) extends naturally to polynomial systems with parameters, under the assumption that the parameters will be interpreted as constants with respect to the derivations and thus will be pushed into the coefficient field. In many cases the `rifsimp` algorithm can perform the analog of a lexicographical Gröbner bases computation, using an appropriately defined ranking.

In some cases, specific values of the parameters will result in systems with exceptional properties. Detecting such systems with non-generic properties is important in applications.

3.5 An example from Robotics

The following example is taken from [3], chapter 6. The Inverse Kinematic problem leads to the following system of polynomial equations:

$$\begin{aligned} &l_3c_1c_2 - l_3s_1s_2 + l_2c_1 - a = 0\\ &l_3c_1s_2 + l_3c_2s_1 + l_2s_1 - b = 0\\ &c_1^2 + s_1^2 - 1 = 0\\ &c_2^2 + s_2^2 - 1 = 0 \end{aligned}$$

where the main variables are c_2,s_2,c_1,s_1, and a,b,l_2,l_3 are viewed as parameters. We are interested in computing a lexicographical Gröbner basis of the system with the ordering

$$c_2 > s_2 > c_1 > s_1$$

on the variables. This actually means that we are computing the Gröbner basis in the polynomial ring $\mathbb{R}(a,b,l_2,l_3)[s_1,s_2,c_1,c_2]$ over the field $\mathbb{R}(a,b,l_2,l_3)$ of rational functions in a,b,l_2,l_3.

Using `PolynomialToPDE` to translate the system to a differential system suitable for processing with **`rif`** we obtain:

$$\begin{gathered} l3\ \frac{\partial^2}{\partial c1\,\partial c2}u - l3\ \frac{\partial^2}{\partial s1\,\partial s2}u + l2\ \frac{\partial}{\partial c1}u - au\\ l3\ \frac{\partial^2}{\partial c1\,\partial s2}u + l3\ \frac{\partial^2}{\partial c2\,\partial s1}u + l2\ \frac{\partial}{\partial s1}u - bu\\ \frac{\partial^2}{\partial {c1}^2}u(c2, s2, c1, s1) + \frac{\partial^2}{\partial {s1}^2}u(c2, s2, c1, s1) - u(c2, s2, c1, s1)\\ \frac{\partial^2}{\partial {c2}^2}u(c2, s2, c1, s1) + \frac{\partial^2}{\partial {s2}^2}u(c2, s2, c1, s1) - u(c2, s2, c1, s1) \end{gathered}$$

Remark. In the first two of the above equations the dependence of u on the independent variables c_2,s_2,c_1,s_1 has been ommited, to save space.

The appropriate ranking is given by the list

$$[[1,0,0,0,0],[0,1,0,0,0],[0,0,1,0,0],[0,0,0,1,0]]$$

which may also be written as a matrix with first row $[1,0,0,0,0]$, etc.

The **`rif`** algorithm takes 22 seconds to compute the reduced form of this system: `DEtools[rifsimp](desys,ranking=[[1,0,0,0,0],[0,1,0,0,0], [0,0,1,0,0],[0,0,0,1,0]],casesplit):`

The algorithm establishes the following 15 cases (in Maple 6) for the parameters a,b,l_2,l_3:

$1 : [l3 \neq 0, al2 \neq 0, -b^2 - a^2 \neq 0]$

$2 : [l3 \neq 0, al2 \neq 0, -b^2 - a^2 = 0, b\,(l3 - l2)\,(l3 + l2) \neq 0]$

$3 : [l3 \neq 0, al2 \neq 0, -b^2 - a^2 = 0, b\,(l3 - l2)\,(l3 + l2) = 0, (l3 - l2)\,(l3 + l2) \neq 0]$

$4 : [l3 \neq 0, al2 \neq 0, -b^2 - a^2 = 0, b\,(l3 - l2)\,(l3 + l2) = 0, (l3 - l2)\,(l3 + l2) = 0, b \neq 0]$

$5 : [l3 \neq 0, al2 \neq 0, -b^2 - a^2 = 0, b\,(l3 - l2)\,(l3 + l2) = 0, (l3 - l2)\,(l3 + l2) = 0, b = 0]$

$6 : [l3 \neq 0, al2 = 0, l2 \neq 0, b \neq 0]$

$7 : [l3 \neq 0, al2 = 0, l2 \neq 0, b = 0, (l3 - l2)\,(l3 + l2) \neq 0]$

$8 : [l3 \neq 0, al2 = 0, l2 \neq 0, b = 0, (l3 - l2)\,(l3 + l2) = 0]$

$9 : [l3 \neq 0, al2 = 0, l2 = 0, -a^2 + l3^2 - b^2 \neq 0]$

$10 : [l3 \neq 0, al2 = 0, l2 = 0, -a^2 + l3^2 - b^2 = 0]$

$11 : [l3 = 0, l2 \neq 0, l2^2 - a^2 - b^2 \neq 0]$

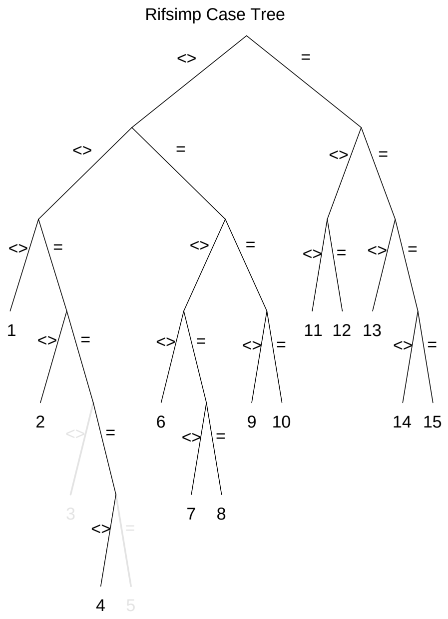
$12 : [l3 = 0, l2 \neq 0, l2^2 - a^2 - b^2 = 0]$

$13 : [l3 = 0, l2 = 0, a \neq 0]$

$14 : [l3 = 0, l2 = 0, a = 0, b \neq 0]$

$15 : [l3 = 0, l2 = 0, a = 0, b = 0]$

In Maple 7, the output leads to 13 cases (since the 2 inconsistent cases Case 3 and Case 5 above are not counted). In Maple 6, the caseplot algorithm produces the following tree:



Acknowledgements

We would especially like to thank Allan Wittkopf for his help on this project.

References

[1] F. Boulier, D. Lazard, F. Ollivier, and M. Petitot, *Representation for the radical of a finitely generated differential ideal*, in Proc. ISSAC 1995, ACM Press, 1995, pp. 158–166.

[2] R. Corless, *Multivariate polynomial equations with multiple zeros solved by matrix eigenproblems*, SIGSAM Bull. **114**, (1995) pp. 1-7.

[3] D. Cox, J. Little, D. O'Shea, *Ideals, Varieties and Algorithms*, Springer, 1997, Second Edition.

[4] I. Kotsireas, *Central Configurations in the Newtonian N-body Problem of Celestial Mechanics*, 2000, submitted to Contemporary Mathematics

[5] E. Mansfield, *Differential Gröbner Bases*, PhD thesis, Univ. of Sydney, 1991. www-address: www.ukc.ac.uk/IMS/maths/people/E.L.Mansfield.html

[6] H.M. Möller, H.J. Stetter, *Multivariate polynomial equations with multiple zeros solved by matrix eigenproblems* Numer. Math. **70**, (1995) pp. 311-329.

[7] G. Reid, A. Wittkopf, and A. Boulton, *Reduction of systems of nonlinear partial differential equations to simplified involutive forms*, European J. of Appl. Math., 7 (1996), pp. 635–666.

[8] C. Rust, *Rankings of Derivatives for Elimination Algorithms and Formal Solvability of Analytic Partial Differential Equations*, PhD thesis, Univ. Chicago, 1998. www-address: www.cecm.sfu.ca/˜rust.

[9] C. Rust, G. Reid, and A. Wittkopf, *Existence and Uniqueness Theorems for Formal Power Series Solutions of Analytic Differential Systems*, in Proc. ISSAC '99, Vancouver, S. Dooley, ed., acm Press, 1999, pp. 105–112.

[10] F. Schwarz, *An algorithm for determining the size of symmetry groups*, Computing, 49 (1992), pp. 95–115.

[11] ——, *Reduction and completion algorithms for Partial Differential Equations*, in Proc. ISSAC '92, Berkeley, P. Wang, ed., acm Press, 1992, pp. 49–56.

[12] ——, *Janet Bases of 2nd Order Differential Equations*, in Proc. ISSAC '96, Zurich, Y. N. Lakshman, ed., acm Press, 1996, pp. 179–188.

[13] W. Seiler, *Analysis and application of the formal theory of partial differential equations*, PhD thesis, Lancaster University, 1994.