Symmetry Classification Using Invariant Moving Frame

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Abstract

A procedure is given for classifying the dimension and structure of symmetry groups leaving invariant individual members of a class of physical differential equations, which exploits the knowledge of an equivalence group of the class. The classes considered contain arbitrary constants or functions. The components of the Lie symmetry vector fields generating symmetries of individual members of such classes satisfy systems of overdetermined partial differential equations. Such *symmetry defining systems* are easily generated by symbolic manipulation algorithms. Such systems can in theory be reduced to simplified form by symbolic implementations of (commutative) differential elimination algorithms and much desirable information can then be obtained. However the reduction is often made prohibitive by the presence of the unspecified functions.

Given the symmetry defining system for the symmetry vector fields, we show how to refer the system to a moving frame. We choose the frame to be covariant under the action of an equivalence group of the class. With the defining system in manifestly covariant form,

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[‡] GJR gratefully acknowledges support from the Natural Sciences and Engineering Research Council of Canada. He also acknowledges many helpful discussions with Elizabeth Mansfield on the topic of this paper, and support of an ESPRC grant from the U.K. government for a visit to the University of Kent in Canterbury. Support from University of Canberra is also acknowledged.

reduction to a non-commutative Riquier Basis is achieved by a non-commutative generalization of the differential elimination procedure originally developed by Riquier. An existence and uniqueness theorem for such non-commutative Riquier Bases given in work by Lemaire, Reid and Zhang, enables us to compute the dimension and structure of the symmetry algebras. Execution of our technique yields a classification tree for the symmetries of the class. The method is applied to a class of nonlinear diffusion convection equations $u_t = ((D(u)u_x)_x - (K(u))_x$ and to a class of second order linear hyperbolic equations $z_{xy} + A(x,y)z_x + B(x,y)z_y + C(x,y)z = 0$ which are both invariant under large easily determined equivalence groups. In these examples the complexity of the calculations is much reduced by the use of moving frames invariant under the given equivalence group.

AMS Classification: 35N10, 58J70, 53A55, 13P10, 12H05.

1 Introduction

Lie symmetries of differential equations are the basis of several useful techniques in applied mathematics, such as finding invariant solutions, solving ordinary differential equations in formula, finding conservation laws and linearizations [25, 21, 4, 22]. Some symmetries are known a priori on physical grounds, for instance an isotropic medium will have rotational symmetries. However, in general a given differential equation must be analyzed to find all its symmetries. The analysis proceeds by seeking vector fields

$$\mathbf{X} = \xi^i(w)\frac{\partial}{\partial w^i}$$

on the space of independent and dependent variables, such that the PDEs are invariant under the action of $\mathbf{X}$. The unknown components ξ^i of the vector fields satisfy a linear homogeneous system of PDEs, the infinitesimal *defining system* for the symmetries. Several programs are available [33, 27, 34] for reducing the defining system and in many cases solving it explicitly and exhibiting structural features of the Lie symmetry algebra. These programs are often successful in performing symmetry analysis for a single differential equation.

However, often one wishes to find the symmetry properties of a *class* of differential equations. For instance, one might be interested in second order ODEs $y'' = f(x, y, y')$, or in wave propagation through an inhomogeneous medium $u_{tt} = c^2(x)u_{xx}$.

Example 1.1. Consider the scalar nonlinear diffusion equation

$$u_t = \big(D(u)u_x\big)_x \tag{1}$$

which is assumed to be nonlinear so that $\dot{D}(u) \neq 0$ where dot denotes differentiation with respect to u. Different functional forms of the diffusivity $D(u)$ will lead to different symmetry groups. The symmetry operator $\mathbf{X} = \xi(x,t,u)\,\partial_x + \tau(x,t,u)\,\partial_t + \eta(x,t,u)\,\partial_u$ has components ξ, τ, η which obey defining equations [25, eq.(6.7.3)]

$$\tau_x = \tau_u = \xi_u = \eta_{uu} = 0 \tag{2a}$$

$$D(2\xi_x - \tau_t) - \dot{D}\eta = 0 \tag{2b}$$

$$D(2\eta_{xu} - \xi_{xx}) + 2\dot{D}\eta_x + \xi_t = 0 \tag{2c}$$

$$D\eta_{xx} - \eta_t = 0. \tag{2d}$$

For given $D(u)$, this is an overdetermined linear homogeneous system.

In analyzing such systems containing arbitrary elements, new problems arise. As the defining equations are manipulated, derivatives of the arbitrary elements (for instance $\dot{D}(u)$, $\ddot{D}(u)$, $\ldots$) accumulate in the coefficients, significantly increasing the algebraic complexity of the equations. One also uncovers *case splittings*, conditional on the arbitrary elements obeying certain *classification equations*. Differentiation and elimination can be used to determine these classification conditions.

Example 1.1 *(cont.).* Equation (2a) implies $\eta_{uu} = 0$, and ssuming that $\dot{D} \neq 0$, condition (2b) implies that $\frac{D}{\dot{D}}(2\xi_x - \tau_t) - \eta = 0$. Thus $(D/\dot{D})^{\ddot{}}\,(2\xi_x - \tau_t) = 0$, where dot denotes differentiation with respect to u. There are two cases:

$$(D/\dot{D})^{\ddot{}} = 0 \tag{3}$$

or $(D/\dot{D})^{\ddot{}} \neq 0$, $2\xi_x - \tau_t = 0$. In this second case, further differentiations and eliminations quickly show that the diffusion equation (1) admits only the obvious symmetries of translation in space and time, and the scaling $(x, t) \mapsto (\alpha x, \alpha^2 t)$ (see [25, §6.7]).

Thus all the cases where the nonlinear diffusion equation (1) has non-obvious symmetries occur for $D(u)$ satisfying the classification condition (3). Such conditions are what investigators aim for in classification problems.

The computational labor for symmetry classification is greater than for symmetry analysis of one system, mainly because algebraic complexity of the defining system can build up explosively as the equations are manipulated. Programs such as [33, 34, 30, 20] can handle classes of PDEs, but can fail due to exhaustion of memory. Even when answers are returned, the case splitting criteria may be so complex as to defy interpretation. To overcome this, various workers [1, 2] follow Ovsiannikov [25] in using *equivalence transformations*, which map equations in a class to other equations in the same class.

Example 1.1 *(cont.).* For the diffusion equation (1), the transformations

$$x = \beta x', \qquad t = t', \qquad u = \gamma u' + \delta, \qquad \beta, \gamma \neq 0 \tag{4}$$

map the diffusion equation to $\gamma u'_{t'} = \gamma\beta^{-2}\big(D(\gamma u' + \delta)u'_{x'}\big)_{x'}$. Hence the coefficient $D(u)$ is mapped to a new coefficient, given by

$$D'(u) = \beta^{-2} D(\gamma u + \delta)$$

When classifying symmetries, the equivalence group is typically used to eliminate parameters from cases.

Example 1.1 *(cont.).* Ovsiannikov [25, §6.7] shows that the nonlinear heat equation (1) admits additional symmetries only if $D(u)$ is a solution of of the classification condition (3), which is easily solved to give the forms:

$$D(u) = a, \qquad D(u) = ae^{mu}, \qquad D(u) = (au+b)^m, \qquad (a, m \neq 0)$$

Parameter elimination by the equivalence group (4) leads to the 'normal forms'

$$D(u) = 1, \qquad D(u) = e^u, \qquad D(u) = u^m, \qquad (m \neq 0).$$

However, this use of equivalence transformations to 'clean up' at the end of a symmetry classification does not address the problem of algebraic complexity which arises in intervening calculations.

In the current paper, we present a method which takes advantage of equivalence transformations at the outset rather than at the end. The goal is to provide a method which has the equivalence group built into it, in the sense that two equations connected by an equivalence transformation will automatically be identified throughout the calculation. Significant clarification and simplification of symmetry classifications are thereby achieved.

Our method is based on differential elimination procedures developed by Reid et al [27, 28, 29, 30] and is based on completion of the defining system by adjoining compatibility conditions [5]. After a finite number of steps the system is obtained in a reduced form where a local existence-uniqueness theorem (originally due to Riquier) can be used to deduce properties of the defining system, such as the dimension of its solution space. The differential elimination method of [27, 18] is capable of systematically exhibiting such symmetry classifying conditions, and similar methods are now employed in several programs [33, 34, 20]. However, these methods are subject to severe expression swell, and often fail for this reason. In addition, the programs often give rise to spurious case splittings: the user pursues first one then another branch, only to find that the branches have no special symmetry properties.

By combining the differential elimination method of Reid with the equivalence group associated with a class of differential equations, we are able to carry out calculations which lessen these computational difficulties. The price paid for these results is twofold. First, we must compute the differential invariants of the equivalence group. Second, the differential elimination and completion must now be performed using a non-commuting basis of differential operators (moving frame). A rigorous non-commutative Gröbner-style theory which gave properties of a class of the output systems is presented in [10]. A different development, allowing nonlinearity not present [10], is given in [15]. That approach yields a non-commutative generalization of Riquier's existence and uniqueness theorem.

The most closely related work to our own is that of Mansfield [19] who gives a method, and an implementation in Maple, of an algorithm which uses the theory of moving frames in Fels and Olver [7, 8]. Such work represents a revitalized interest in Cartan's method of moving frames, its applications and generalizations. For example see [22], [14] and the review paper [24], and also [6] for applications to computer vision. Applications to group invariant numerical integrators are given in [23] in the newly developing area of Geometric Integration [11].

The remainder of this paper is organized as follows. In Section 2 we discuss the adaptation of differential elimination methods to moving frames. Using the equivalence group to find suitable frames is discussed in Section 3 and applied to the example of the nonlinear diffusion equation (1) in Section 4. The calculation of structure constants from a frame defining system is discussed in Section 5. Substantial examples of symmetry classifications are presented in Sections 6 (nonlinear diffusion convection equations) and 7 (linear hyperbolic equations).

2 Frames and Non-Commutative Riquier Basis Form

2.1 Moving frame and structure relations

Differential elimination methods based on Riquier theory and its offspring [13, 27, 30, 5] are stated in a coordinate system, with commuting partial derivative operators. We now relax this and permit an arbitrary basis for first order differential operators.

Definition 2.1. Let W be a neighbourhood in $\mathbb{R}^n$. A moving frame of differential operators on W (hence-forth abbreviated *moving frame*) is a set of n smooth vector fields Δ_1, Δ_2, …, Δ_n which are linearly independent at each point in W.

This differs from the definition of Fels and Olver [7, 8] in which a moving frame invariant under a Lie group $\mathcal{G}$ is a $\mathcal{G}$-invariant map from a manifold to a group. If W has coordinates w, any moving frame can ultimately be referred to the coordinate frame ∂_{w^1}, ∂_{w^2}, …, ∂_{w^n}:

$$\Delta_i = A_i^j(w)\partial_{w^j}. \tag{5}$$

The $n \times n$ matrix $A_i^j(w)$ of smooth functions is to be nonsingular at every point $w \in W$. More generally, given a frame $\{\Delta_i\}_{i=1}^n$, we may change frame to

$$\Delta'_i = A_i^j(w)\Delta_j \tag{6}$$

where the change of frame matrix $A_i^j(w)$ is nonsingular and smooth in W. We denote frame operators by upper case Greek letters Δ, Λ, etc. Moving frames are widely used in geometry; see [35, Vol. II, §7] or most other modern differential geometry texts for details.

Any vector field $\mathbf{Y}$ may be referred to a frame Δ, as $\mathbf{Y} = \theta^i\Delta_i$. Under the change of frame (6), the components θ^i of the vector field transform via

$$\theta^j = A_i^j(w)\theta'^i. \tag{7}$$

Example 2.2. The following frame arises from the nonlinear diffusion convection potential system discussed in §6. Let $\{\partial_v, \partial_x, \partial_t, \partial_u\}$ be a coordinate frame on $\mathbb{R}^4(v, x, t, u)$. Introduce the moving frame Δ given by

$$\begin{aligned} &\Delta_1 = \partial_v && \Delta_2 = \partial_x \\ &\Delta_3 = \partial_t + \dot{K}(u)\partial_x + \big(u\dot{K}(u) - K(u)\big)\partial_v && \Delta_4 = \partial_u \end{aligned} \tag{8}$$

where $K(u)$ is some smooth function. The change of frame matrix has determinant 1, so Δ is indeed a moving frame on $\mathbb{R}^4$.

Let

$$\mathbf{Y} = \chi\partial_v + \xi\partial_x + \tau\partial_t + \eta\partial_u$$

be an arbitrary vector field, with χ, ξ, τ, η functions of (v, x, t, u). Resolving with respect to the moving frame Δ, i.e. $\mathbf{Y} = \theta^i\Delta_i$ gives

$$\begin{aligned} &\theta^1 = \chi - \big(u\dot{K}(u) - K(u)\big)\tau && \theta^2 = \xi - \dot{K}(u)\tau \\ &\theta^3 = \tau && \theta^4 = \eta. \end{aligned} \tag{9}$$

The commutator $[\,\Delta_i, \Delta_j\,]$ of two vector fields Δ_i, Δ_j from a moving frame must be expressible as a linear combination of Δ_k at each point:

$$[\,\Delta_i, \Delta_j\,] = \gamma_{ij}^k \Delta_k, \tag{10}$$

where the *structure functions* γ_{ij}^k of the frame are functions of w, and $\gamma_{ij}^k = -\gamma_{ji}^k$. Relations (10) are the *structure relations* for frame Δ.

2.2 Non-Commutative Riquier Reduction Method

Reid [27, 30] described algorithms for bringing a linear homogeneous system of partial differential equations to a reduced involutive form, whose compatibility conditions yield no new relations. Now suppose we are given a linear PDE system referred to a frame. We seek to adapt the methods to this case, allowing for non-commuting frame operators, successively defining orthonomic, reduced orthonomic, and reduced involutive systems with respect to a frame. These concepts are direct adaptations from the Riquier-Janet theory [13, 37, 27] for systems of differential equations.

Let $\{\Delta_i\}_{i=1}^n$ be a moving frame on $W \subseteq \mathbb{R}^n$, with structure relations $[\Delta_i, \Delta_j] = \gamma_{ij}^k \Delta_k$. Let $\{\theta^i\}_{i=1}^\mu$ be certain dependent variables. If $J = (j_1 j_2 \cdots j_l)$, where each $1 \leq j_i \leq n$, then we use comma notation to indicate the l-th order frame derivative with respect to J:

$$\theta^i_{,J} \equiv \Delta_{j_l} \cdots \Delta_{j_2} \Delta_{j_1} \theta^i$$

The notation $[J]$ will be used to indicate the *symmetric* multi-index associated with J. Let I and J be two multi-indices of orders p_1, p_2 respectively, with $p_1 \leq p_2$. We say $I \subseteq J$ if there exists a multi-index L such that $[J] = [IL]$. Thus $(133) \subseteq (3131)$. If $I \subseteq J$, then we say that $\theta_{,J}$ is a derivative of $\theta_{,I}$. Thus θ_{331} is a derivative of $\theta_{,13}$, differing from $\theta_{,133}$ by terms of order lower than 3. Note that if $[I] = [J]$ so that I is a permutation of J, then $\theta_{,J}$ is a 0-th order derivative of $\theta_{,I}$.

Differential elimination methods rely on an ranking of the partial derivatives u^i_J in the PDEs. For compatibility with the ordinary case, the structure relations can be used to reorder the indices in J to some chosen order. For instance, one could insist wherever a frame derivative $\theta^i_{,J}$ appears, that the indices in J be non-decreasing. Using the structure relations to effect this permutation gives rise at worst to terms of order lower than $|J|$.

Definition 2.3. A *ranking* of frame derivatives is a linear order relation $\prec$ on frame derivatives $\theta^j_{,J}$, with the properties:

1. (Transitivity) If $\Delta_{[I]}\theta^i \prec \Delta_{[J]}\theta^j$ and $\Delta_{[J]}\theta^j \prec \Delta_{[K]}\theta^k$, then $\Delta_{[I]}\theta^i \prec \Delta_{[K]}\theta^k$.
2. (Trichotomy) If $\Delta_{[I]}\theta^i$, $\Delta_{[J]}\theta^j$ are two derivatives, exactly one of (a) $\Delta_{[I]}\theta^i \prec \Delta_{[J]}\theta^j$, (b) $\Delta_{[J]}\theta^j \prec \Delta_{[I]}\theta^i$, (c) $\Delta_{[J]}\theta^j = \Delta_{[I]}\theta^i$, is true.
3. (Stability under differentiation) If $\Delta_{[I]}\theta^i \prec \Delta_{[J]}\theta^j$, then $\Delta_{[IL]}\theta^i \prec \Delta_{[JL]}\theta^j$ for all arbitrary order multi-indices L.
4. (Least element) $\Delta_{[J]}\theta^j \prec \Delta_{[JL]}\theta^j$ for all nonempty multi-indices L.

As an example of a ranking, consider the graded lexicographic ranking $\theta^i_{,I} \prec \theta^j_{,J}$ if

1. $|I| < |J|$
2. $|I| = |J|$, but $i < j$
3. $|I| = |J|$, $i = j$ but the first nonzero member of the sequence

$$N_1(I) - N_1(J), \quad N_2(I) - N_2(J), \quad \ldots, \quad N_n(I) - N_n(J)$$

is negative.

Here the notation $N_i(J)$ represents the count of i's in the multi-index J.

Example 2.4. Suppose there are two dependent variables θ^1, θ^2, and two frame operators Δ_1, Δ_2, and that all multi-indices J are arranged to be nonincreasing. Then graded lexicographic order is

$$\theta^1 \prec \theta^2 \prec \theta^1_{,1} \prec \theta^1_{,2} \prec \theta^2_{,1} \prec \theta^2_{,2} \prec \theta^1_{,11} \prec \theta^1_{,12} \prec \theta^1_{,22} \prec \theta^2_{,11} \prec \theta^2_{,12} \prec \theta^2_{,22} \prec \cdots$$

With ranking defined, we presume available procedures for the following tasks:

maxorder(R)
  input: A finite set R of frame derivatives $\theta^j_{,J}$
  output: The element $r \in R$ highest in the ranking

removepermutations(R)

input: A set R of frame equations.
action: For each equation $r \in R$, check for presence in r of derivatives $\theta^j_{,I}$, not in nondecreasing order.
If present, use structure relations to write them all in nondecreasing order.
output: Equations R with permutations removed.

leadingderiv(*eqn*)
input: A frame equation *eqn*.
output: The derivative of highest order occurring in *eqn*:
$leadingderiv := maxorder\{\theta^j_{,J} \mid \theta^j_{,J} \text{ present in } eqn\}$.

solve(*eqn*,*deriv*)
input: A frame equation *eqn*.
A frame derivative $deriv = \theta^j_{,J}$ occurring in *eqn*.
output: *eqn* rewritten in the form $\theta^j_{,J} = \text{rhs}$.

subst(*eqn*,*U*)
input: An equation *eqn* in solved form $\Delta_J\theta^j = \text{rhs}$.
A set U of frame equations.
action: Replace each occurrence of $\Delta_I\theta^j$ where I is a permutation of J by $\Delta_I\theta^j = \text{rhs} + \text{permutation terms}$
output: U with $\Delta_J\theta^j$ substituted out.

We adapt the concept of orthonomic system [26, 13] to frame systems.

Definition 2.5. A linear homogeneous frame system is in *orthonomic form* if

(i) Each equation is resolved in the form
$$\theta^i_{,I} = f^J_j$$
where each f^J_j is a function of terms lower in the ranking than $\theta^i_{,I}$.

(ii) $\theta^i_{,I}$ is strictly higher in the ranking than any terms $\theta^j_{,J}$ on the right hand side.

(iii) A given derivative $\theta^j_{,J}$ cannot appear in both the left and right hand sides of the same equation.

The highest order derivative occurring in an equation will be called the *leading* derivative.

Algorithm 2.6. (orthonomic)

function *orthonomic*(*system*)
    unsolved := *system*
    solved := $\emptyset$
    repeat
        unsolved := *removepermutations*(*unsolved*)
        $maxderiv := maxorder\{leadingderiv(eqn) \mid eqn \in U\}$
        $maxset := \{eqn \in unsolved \mid leadingderiv(eqn) = maxderiv\}$
        nexteqn := (any element in *maxset*)
        nexteqn := *solve*(*nexteqn*,*maxderiv*)
        $solved := subst(nexteqn,solved) \cup \{nexteqn\}$
        $unsolved := subst(nexteqn,unsolved \setminus \{nexteqn\})$
    until $unsolved = \emptyset$
    return(*solved*)
end

This algorithm must terminate after finitely many steps by a standard Noetherian argument.

Definition 2.7. A frame system is in *reduced* orthonomic form if it is orthonomic (Definition 2.5) and in addition

(iv) No frame derivative in the system is the derivative of any frame derivative on the left hand side.

Note that since $\theta_{,ij}$ is regarded as a derivative of $\theta_{,j}$, a system with both $\theta_{,j}$ and $\theta_{,ij}$ on the left hand side would not be reduced orthonomic. A system can be brought to reduced orthonomic form as follows:

Algorithm 2.8. (reduce)

function *reduce*(*system*)
    repeat
        system := *orthonomic*(*system*)
        while exist $\theta^j_{,J}$, derivative of leading $\Delta_I\theta^j$ **do**
            system := *reduce* ($\theta^j_{,I}$, $\theta^j_{,J}$, *system*)
        end do
    until *system* is orthonomic
    return(*system*)
end

Let a frame system be given in reduced orthonomic form. Suppose the system contains the two equations

$$\theta^i_{,I} = \text{rhs}_1, \qquad \theta^i_{,J} = \text{rhs}_2 \tag{11}$$

Define the 'least common multiple' [lcm(I, J)] of multi-indices I, J by

$$N_j[\text{lcm}(I,J)] = \max_j\{N_j(I), N_j(J)\}.$$

and suppose that the indices are ordered e.g. to be nonincreasing. Then $\text{lcm}(I, J)$ is the smallest multi-index K such that $\theta_{,K}$ is a derivative of both $\theta_{,I}$ and $\theta_{,J}$.

Example 2.9. Let $I = (11233)$, and $J = (122)$. The 'lcm' of I, J is (112233).

Suppose $[K] = [IL] = [JM]$ for some multi-indices L, M. Then

$$\begin{aligned}\theta^i_{,K} &= \theta^i_{,IL} + \text{permutation terms}\\ &= (\text{rhs}_1)_{,L} + \text{permutation terms}\end{aligned}$$

and

$$\begin{aligned}\theta^i_{,K} &= \theta^i_{,JM} + \text{permutation terms}\\ &= (\text{rhs}_2)_{,M} + \text{permutation terms.}\end{aligned}$$

Equating these two expressions yields the *compatibility condition*

$$(\text{rhs}_1)_{,L} - (\text{rhs}_2)_{,M} + \text{permutation terms} = 0$$

of (11). This expression can then be reduced modulo the original reduced orthonomic system.

If compatibility conditions are adjoined to a reduced orthonomic frame system, the composite system is no longer in solved form, and must be brought once again to reduced orthonomic form. We distinguish systems where this process does not lead to addition of further relations.

Definition 2.10. A reduced orthonomic frame system R is in the form of an (auto-reduced) non-commutative Riquier Basis [15] if the compatibility conditions of R become trivial after reduction mod R.

A frame system may be brought to such a form by putting it into reduced orthonomic form, appending compatibility conditions, then repeating the process:

Algorithm 2.11. (Non-Commutative Riquier Basis)

function *Non-Commutative-Riquier-Basis*(*system*)
  repeat
    system := *reduce*(*system*)
    compat := *compatibility*(*system*)
    system := *system* $\cup$ *compat*
  until *compat* $= \emptyset$
  return *system*
end

3 Invariant Form of Classification and Properties of Non-Commutative Riquier Bases

Associated with a system in the form of a non-commutative Riquier Basis are two disjoint sets of frame derivatives.

Definition 3.1. If $\theta^j_{,J}$ occurs on the left hand side, or is a derivative of $\theta^j_{,I}$ occurring on the left hand side of a frame system in the form of a non-commutative Riquier Basis, then it is called a *principal* derivative of the system. If $\theta^j_{,J}$ is not principal it is called a *parametric* derivative.

Note that the criterion for whether a frame derivative is principal (and hence also for parametric) respects permutation of the multi-index J defining it. Thus it is not possible to have $\theta_{,12}$ principal and $\theta_{,21}$ parametric.

The importance of systems of PDEs in the form of a Riquier Basis is that there is available a theory for existence of a unique solution in the neighbourhood of initial data obtained by specifying each parametric derivative at a point [37],[32]. For example if S is a linear homogeneous system in Riquier Basis form, with independent variables w and dependent variables θ, written in a *coordinate* frame. Let w_0 be a point at which the coefficient functions are analytic. If the parametric derivatives of S are of finite number r, and values of these parametric derivatives are specified at w_0, there exists a unique analytic solution of S in a neighbourhood of w_0 satisfying these initial conditions. In particular the system has an r-dimensional solution space. If the parametric derivatives are not finite in number the system has an infinite-dimensional solution space.

Briefly put, the solution space dimension of such a system is equal to the number of parametric derivatives in the system (see [15] for a discussion of the non-commutative case).

In its full generality, the theory is not restricted to linear systems, and a careful enumeration of initial data sufficient to guarantee existence and uniqueness for the infinite dimensional case is also performed. See [32, 31, 30, 5] gives details, examples and computational algorithms.

3.1 Classification Procedure

In [28] it is shown that classification can be performed *without* solving the defining equations. The method algorithmically finds classification conditions by appending compatibility conditions to the defining system (also see Mansfield [18] and Boulier et al [5]).

To effect a classification for a frame system containing arbitrary elements, the procedures detailed in §2.2 are slightly modified:

1. A classifying (frame) system C is now present. This specifies equations to be satisfied by the arbitrary elements in the differential equations, and must be reduced to non-commutative Riquier Basis form each time a new equation is added to it.

2. Whenever the defining system S is modified by a differentiation operation, it is reduced modulo the classifying system C.

3. The reduction to orthonomic form will only complete if all divisions were unequivocally possible (e.g., the coefficients are constants or are known to be nonzero by virtue of inequalities I). If any other division arises, note the *pivot* which is the coefficient of the leading derivative. The calculation then splits into two cases:

    - Adjoin $\mathit{pivot} = 0$ to the list of classifying equations.
    - Adjoin $\mathit{pivot} \neq 0$ to the list of classifying inequalities.

    Each case is pursued separately (see [5] for an algebraic interpretation).

A *tree* of possibilities is thereby built up, accumulating a *classifying system* C—consisting of the original frame system C satisfied by the arbitrary elements—and a set I of classifying *frame inequalities* which result from demanding that various pivots *not* vanish.

3.2 Invariant form of group classification

We can now perform symmetry classification of a class of differential equations in an invariant manner, i.e., with each step being invariant with respect to the action of the equivalence group of the class. The above classification method is simply carried out referred to a frame invariant under the action of the equivalence group. The steps are:

1. Derive the equivalence group of the differential equations.

2. Derive defining equations for the symmetries of the differential equations.

3. Construct invariants and invariant augmented frame(s) of the equivalence group, along with their structure relations. (Different frames may be necessary for different arbitrary elements.)

4. Rewrite the defining system in terms of the invariant frame, with invariant coefficients.

5. Rewrite the classifying system in terms of invariants and frame operators.

6. Invoke the classification procedure with the given classifying system and with no classifying inequalities initially.

7. For each leaf of the resulting tree there is a non-commutative Riquier Basis form of the defining system: find the size and structure of the associated Lie symmetry algebra (§5).

This method for symmetry classification, which first appeared in Lisle's thesis [17], is therefore a generalization of [28] to a case where an equivalence group is available. Equivalence information is built into the method through the invariant frame. In many cases completion to a non-commutative Riquier Basis can be achieved by hand, even for systems requiring large amounts of computer time and memory. This is presumably because much symmetry information is in the equivalence group. Factoring out this information reduces the computational complexity. A useful feature of our method is that, since it is expressed in terms of invariants of the equivalence group, the case splittings involved are likewise invariant. This means that two equations connected by an equivalence transformation must end up on the same branch of the classification tree. This can dramatically reduce the number of spurious case splittings generated by the method, with consequent gains in interpretability of the tree.

4 Application to Nonlinear Diffusion Equation

Before exhibiting two substantial classifications (§§6,7), we give a simpler example. Consider the nonlinear diffusion equation

$$u_t = \big(D(u)u_x\big)_x \tag{12}$$

We seek to write the defining equations (2) for its symmetries [25, eq.(6.7.3)] in a form invariant under the action of the six-parameter equivalence group:

$$(a)\begin{cases} x' &= \alpha x + a \\ t' &= \alpha^2 t + b \\ u' &= u \\ D' &= D \end{cases} \qquad (b)\begin{cases} x' &= \beta x \\ t' &= t \\ u' &= \gamma u + \delta \\ D' &= \beta^2 D \end{cases} \tag{13}$$

Here transformations (a) are the symmetries (self-equivalences) common to the class, and (b) are the (non-self) equivalence transformations.

We assume outright that $D > 0$, and that $\dot{D} \neq 0$, leaving aside the linear heat equation. Introduce the frame and infinitesimals

$$\begin{aligned} \Delta_1 &:= D^{1/2}\,\partial_x & \theta^1 &:= D^{-1/2}\xi \\ \Delta_2 &:= \partial_t & \theta^2 &:= \tau \\ \Delta_3 &:= D/\dot{D}\,\partial_u & \theta^3 &:= \dot{D}/D\,\eta \end{aligned} \tag{14}$$

and get the scalar invariant

$$J := \frac{D\ddot{D}}{\dot{D}^2} - 1$$

which satisfies relations

$$J_{,1} = 0, \qquad J_{,2} = 0. \tag{15}$$

The only nontrivial structure relation for Δ is

$$[\Delta_1, \Delta_3] = -\tfrac{1}{2}\Delta_1. \tag{16}$$

The defining system (2) becomes

$$\begin{array}{lll} \theta^1_{,3} = -\frac{1}{2}\theta^1 & \theta^2_{,1} = 0 & \theta^3_{,11} = \theta^3_{,2} \\ & \theta^2_{,2} = 2\theta^1_{,1} - \theta^3 & \theta^3_{,31} = \frac{1}{2}\theta^1_{,11} + (J-1)\theta^3_{,1} - \frac{1}{2}\theta^1_{,2} \\ & \theta^2_{,3} = 0 & \theta^3_{,33} = J\theta^3_{,3} + J_{,3}\theta^3 \end{array} \tag{17}$$

Now that the frame Δ has been introduced, we can apply the frame reduction method to (17). Compute compatibility, for instance

$$(\theta^2_{,3})_{,2} - (\theta^2_{,2})_{,3} = -(2\theta^1_{,1} - \theta^3)_{,3}.$$

Reducing the left hand side mod structure relations (16), this becomes $0 = 2\theta^1_{,13} - \theta^3_{,3}$. Reduction mod the defining system (17) gives $\theta^3_{,3} = 0$, which is adjoined to the system. Subsequent reduction gives $J_{,3}\theta^3 = 0$, giving the case splitting $J_{,3} = 0$, $J_{,3} \neq 0$.

If $J_{,3} \neq 0$, then $\theta^3 = 0$, and the system quickly gives a non-commutative Riquier Basis form with three-dimensional solution space. This is the generic case of three symmetries ∂_x, ∂_t, $x\partial_x + 2t\partial_t$. However, if $J_{,3} = 0$ (so that J is a constant), further compatibility conditions and reductions bring the system to

$$\begin{array}{lll} \theta^1_{,11} = 2(1-J)\theta^3_{,1} & \theta^2_{,1} = 0 & \theta^3_{,11} = 0 \\ \theta^1_{,2} = 0 & \theta^2_{,2} = 2\theta^1_{,1} - \theta^3 & \theta^3_{,2} = 0 \\ \theta^1_{,3} = -\frac{1}{2}\theta^1 & \theta^2_{,3} = 0 & \theta^3_{,3} = 0 \end{array} \tag{18}$$

along with $(3-4J)\theta^3_{,1} = 0$, and the pivot $(3-4J)$ gives a case splitting.

If $J \neq 3/4$, then $\theta^3_{,1} = 0$, and the system has non-commutative Riquier Basis form

$$\begin{array}{lll} \theta^1_{,11} = 0 & \theta^2_{,1} = 0 & \theta^3_{,1} = 0 \\ \theta^1_{,2} = 0 & \theta^2_{,2} = 2\theta^1_{,1} - \theta^3 & \theta^3_{,2} = 0 \\ \theta^1_{,3} = -\frac{1}{2}\theta^1 & \theta^2_{,3} = 0 & \theta^3_{,3} = 0. \end{array}$$

There are four parametric derivatives θ^1, θ^2, θ^3, $\theta^1_{,1}$, so the symmetry algebra is of dimension four.

If $J = 3/4$, system (18) is in the form of a non-commutative Riquier Basis. The parametric derivatives θ^1, θ^2, θ^3, $\theta^1_{,1}$, $\theta^3_{,1}$, give a five-dimensional symmetry algebra.

5 Computation of Structure Constants

Consider a defining system for the components of a Lie algebra of vector fields. A finite-dimensional Lie algebra is characterized up to isomorphism by structure constants C^k_{ij}. We show how to find C^k_{ij} directly from the (frame) defining system without solving the equations. The method is a generalization to the non-commuting case of the method of Reid et al. [29].

5.1 Initial Data

Suppose a defining system has been completed to non-commutative Riquier Basis form, and is of order $K \geq 1$. (If $K = 0$, the solution space is either trivial or ∞-dimensional). Indeed, we may assume that the system is in a form where each equation is solved for its highest order derivative:

$$\mathrm{FramePrinc}_{\leq}(\Theta_K) = \mathbf{A}(w)\,\mathrm{FramePar}_{\leq}(\Theta_{K-1}) \tag{19}$$

where Θ_K represents frame derivatives of order up to K:

$$\Theta_K = \left\{\theta^i_{,J} \mid i = 1, \ldots, n;\ 0 \leq |J| \leq K\right\}$$

where, as usual, J is a multi-index. The solution space is assumed finite-dimensional, so every frame derivative of order exactly K is principal, and all parametric frame derivatives are of order $K-1$ or lower.

The coefficient matrix $\mathbf{A}(w)$ varies from point to point. Let w_0 be a regular point for the defining system, so that $\mathbf{A}$ is smooth at w_0. At w_0, the set of values of $\mathrm{FramePar}_{\leq}(\Theta_{K-1})$ is a vector space, which we call *initial data space*. According to the formal Riquier existence and uniqueness Theorem [15] each specification of initial data gives rise to a unique formal power series solution of the defining system. Hence there is a vector space isomorphism $\mathcal{S}$ from the space of initial data at w_0 to the space of formal power series solutions at w_0.

5.2 Commutator bracket

The formal solutions are components of a vector field, and with the usual commutator bracket on vector fields, the solutions at w_0 are therefore also a Lie algebra. The commutator bracket on solutions can be used to induce a bracket on initial data. Let $\mathbf{a}$, $\mathbf{b}$ be initial data vectors. Define

$$[\mathbf{a}, \mathbf{b}]_{\text{ID}} := \mathcal{S}^{-1}\big([\mathcal{S}(\mathbf{a}), \mathcal{S}(\mathbf{b})]\big)$$

That is, the commutator of two initial data vectors is found by constructing the associated formal solutions, taking commutators, then evaluating the initial data of the commutator vector field.

Let the (unique) formal solutions associated with $\mathbf{a}, \mathbf{b}$ be ϕ, ψ: $\phi = \phi^i \Delta_i$, $\psi = \psi^i \Delta_i$. Thus by definition

$$\text{FramePar}_{\leq}(\phi)(w_0) = \mathbf{a}, \qquad \text{FramePar}_{\leq}(\psi)(w_0) = \mathbf{b}.$$

The commutator is

$$\omega = [\phi, \psi] = (\phi^i \psi^k_{,i} - \psi^i \phi^k_{,i})\Delta_k + \phi^i \psi^j [\Delta_i, \Delta_j] \tag{20}$$

If the frame Δ has structure relations $[\Delta_i, \Delta_j] = \gamma^k_{ij}\Delta_k$, the k-th component of the commutator is

$$\omega^k = \phi^i \psi^k_{,i} - \psi^i \phi^k_{,i} + \gamma^k_{ij}\phi^i \psi^j$$

Let the initial data associated with ω be $\mathbf{c}$, that is,

$$\mathbf{c} = \text{FramePar}_{\leq}(\omega)(w_0)$$

Each component of $\mathbf{c}$ is therefore $(\omega^k)_{,J}(w_0)$ for some k, J. Now, the expression

$$\big(\phi^i \psi^k_{,i} - \psi^i \phi^k_{,i} + \gamma^k_{ij}\phi^i \psi^j\big)_{,J}$$

can be evaluated using Leibniz rule to an expression involving derivatives of ϕ, ψ up to order K, then reduced modulo the defining system (19) to an expression involving only $\text{FramePar}_{\leq}(\phi)$, $\text{FramePar}_{\leq}(\psi)$. Evaluation at w_0 gives an expression involving only $\mathbf{a}, \mathbf{b}$. By doing so, each component of the commutator initial data $\mathbf{c}$ is expressed as a skew-symmetric bilinear function of $\mathbf{a}, \mathbf{b}$:

$$c^k = B^k_{ij}(w_0) a^i b^j$$

where $B_{ij} = -B_{ji}$. By construction, the quantities $B^k_{ij}(w_0)$ are structure constants of the Lie algebra induced on initial data space, and therefore of the desired Lie algebra of vector fields. Thus C^k_{ij} is the k-th piece of initial data for the commutator of solution i with solution j.

Example 5.1. Consider the defining system (18) with $J = 3/4$, which is in the form of a non-commutative Riquier Basis. The θ^i are components of a vector field $\mathbf{Y} = \theta^i \Delta_i$ referred to a frame $\{\Delta_1, \Delta_2, \Delta_3\}$ with nontrivial structure relation (16). The parametric derivatives are θ^1, θ^2, θ^3, $\theta^1_{,1}$, $\theta^3_{,1}$. Let

$$\phi = \phi^1\Delta_1 + \phi^2\Delta_2 + \phi^3\Delta_3, \qquad \psi = \psi^1\Delta_1 + \psi^2\Delta_2 + \psi^3\Delta_3,$$

be two solutions, and let $\omega = [\phi, \psi]$. Taking commutators (20) and using structure relation (16), gives for instance

$$\omega^1 = (\phi^1\psi^1_{,1} - \psi^1\phi^1_{,1}) + (\phi^2\psi^1_{,2} - \psi^2\phi^1_{,2}) + (\phi^3\psi^1_{,3} - \psi^3\phi^1_{,3}) - \tfrac{1}{2}(\phi^1\psi^3 - \psi^1\phi^3)$$

After substituting out principal derivatives from (18), we obtain

$$\begin{aligned}
\omega^1 &= \phi^1\psi^1_{,1} - \psi^1\phi^1_{,1} \\
\omega^2 &= 2(\phi^2\psi^1_{,1} - \psi^2\phi^1_{,1}) - (\phi^2\psi^3 - \psi^2\phi^3) \\
\omega^3 &= \phi^1\psi^3_{,1} - \psi^1\phi^3_{,1})
\end{aligned}$$

Differentiating and again reducing mod (18) gives

$$\begin{aligned}
\omega^1_{,1} &= \tfrac{1}{2}(\phi^1\psi^3_{,1} - \psi^1\phi^3_{,1}) \\
\omega^3_{,1} &= \phi^1_{,1}\psi^3_{,1} - \psi^1_{,1}\phi^3_{,1})
\end{aligned}$$

Evaluating at the initial data point gives

$$\begin{aligned}
c^1 &= a^1b^4 - b^1a^4 && (C^1_{14} = 1) \\
c^2 &= 2(a^2b^4 - b^2a^4) - (a^2b^3 - b^2a^3) && (C^2_{24} = 2,\ C^2_{23} = -1) \\
c^3 &= a^1b^5 - b^1a^5 && (C^3_{15} = 2) \\
c^4 &= \tfrac{1}{2}(a^1b^5 - b^1a^5) && (C^4_{15} = \tfrac{1}{2}) \\
c^5 &= a^4b^5 - b^4a^5 && (C^5_{45} = 1)
\end{aligned}$$

The Lie algebra therefore has commutation relations

$$\begin{aligned}
&[\mathbf{Y}_1, \mathbf{Y}_4] = \mathbf{Y}_1 \qquad [\mathbf{Y}_1, \mathbf{Y}_5] = \mathbf{Y}_3 + \tfrac{1}{2}\mathbf{Y}_4 \qquad [\mathbf{Y}_2, \mathbf{Y}_3] = -\mathbf{Y}_2 \\
&[\mathbf{Y}_2, \mathbf{Y}_4] = 2\mathbf{Y}_2 \qquad [\mathbf{Y}_4, \mathbf{Y}_5] = \mathbf{Y}_5
\end{aligned}$$

Note that it was not necessary to explicitly construct solutions during this process.

6 Potential Diffusion Convection System

We now present a substantial computational example, applying the frame method to the diffusion convection potential system

$$v_x = u, \qquad v_t = Du_x - K. \tag{21}$$

The arbitrary elements $D(u)$ (diffusivity) and $K(u)$ (convection) obey the auxiliary system

$$\begin{aligned} D_v &= D_x = D_t = 0 \\ K_v &= K_x = K_t = 0 \end{aligned} \tag{22}$$

It is assumed at the outset that $D > 0$; this condition is preserved by the transformations below.

6.1 Equivalence group

A calculation detailed in [17] shows that the class of equations (21) admits the equivalence operators

$$\begin{array}{ll} \mathbf{X}_1 = \partial_v & \mathbf{X}_6 = x\,\partial_v + \partial_u \\ \mathbf{X}_2 = \partial_x & \mathbf{X}_7 = \frac{1}{2}v\,\partial_v - \frac{1}{2}x\,\partial_x + u\,\partial_u + \frac{1}{2}K\,\partial_K - D\,\partial_D \\ \mathbf{X}_3 = \partial_t & \mathbf{X}_8 = -v\,\partial_x + u^2\,\partial_u + uK\,\partial_K - 2uD\,\partial_D \\ \mathbf{X}_4 = -t\,\partial_v + \partial_K & \mathbf{X}_9 = -v\,\partial_v - x\,\partial_x - 2t\,\partial_t + K\,\partial_K \\ \mathbf{X}_5 = t\,\partial_x + u\,\partial_K & \mathbf{X}_{10} = v\,\partial_v + x\,\partial_x + t\,\partial_t + D\,\partial_D \end{array} \tag{23}$$

The commutation relations of the algebra are

$$\begin{array}{llll} [\mathbf{X}_1, \mathbf{X}_7] = \frac{1}{2}\mathbf{X}_1 & [\mathbf{X}_1, \mathbf{X}_8] = -\mathbf{X}_2 & [\mathbf{X}_1, \mathbf{X}_9] = -\mathbf{X}_1 & [\mathbf{X}_1, \mathbf{X}_{10}] = \mathbf{X}_1 \\ [\mathbf{X}_2, \mathbf{X}_6] = \mathbf{X}_1 & [\mathbf{X}_2, \mathbf{X}_7] = -\frac{1}{2}\mathbf{X}_2 & [\mathbf{X}_2, \mathbf{X}_9] = -\mathbf{X}_2 & [\mathbf{X}_2, \mathbf{X}_{10}] = \mathbf{X}_2 \\ [\mathbf{X}_3, \mathbf{X}_4] = -\mathbf{X}_1 & [\mathbf{X}_3, \mathbf{X}_5] = \mathbf{X}_2 & [\mathbf{X}_3, \mathbf{X}_9] = -2\mathbf{X}_3 & [\mathbf{X}_3, \mathbf{X}_{10}] = \mathbf{X}_3 \\ [\mathbf{X}_4, \mathbf{X}_7] = \frac{1}{2}\mathbf{X}_4 & [\mathbf{X}_4, \mathbf{X}_8] = \mathbf{X}_5 & [\mathbf{X}_4, \mathbf{X}_9] = \mathbf{X}_4 & \\ [\mathbf{X}_5, \mathbf{X}_6] = -\mathbf{X}_4 & [\mathbf{X}_5, \mathbf{X}_7] = -\frac{1}{2}\mathbf{X}_5 & [\mathbf{X}_5, \mathbf{X}_9] = \mathbf{X}_5 & \\ [\mathbf{X}_6, \mathbf{X}_7] = \mathbf{X}_6 & [\mathbf{X}_6, \mathbf{X}_8] = 2\mathbf{X}_7 & [\mathbf{X}_7, \mathbf{X}_8] = \mathbf{X}_8 & \end{array}$$

This 10-dimensional algebra $\mathcal{L}$ is the semidirect sum of the following subalgebras

$$\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\} \oplus_s \{\mathbf{X}_4, \mathbf{X}_5\} \oplus_s \{\mathbf{X}_6, \mathbf{X}_7, \mathbf{X}_8\} \oplus_s \{\mathbf{X}_9\} \oplus_s \{\mathbf{X}_{10}\}$$

These subalgebras correspond to generating the equivalence group from the following subgroups:

$$
(a)\begin{cases}\bar v &= v+\varepsilon_1\\ \bar x &= x+\varepsilon_2\\ \bar t &= x+\varepsilon_3\end{cases}\quad
(b)\begin{cases}\bar v &= v-\kappa_1 t\\ \bar x &= x+\kappa_2 t\\ \bar K &= K+\kappa_1+\kappa_2 u\end{cases}\quad
(c)\begin{cases}\bar v &= \alpha v+\beta x\\ \bar x &= \gamma v+\delta x\\ \bar u &= \frac{\alpha u+\beta}{\gamma u+\delta}\\ \bar K &= \frac{K}{\gamma u+\delta}\\ \bar D &= (\gamma u+\delta)^2 D\end{cases}
$$

$$
(d)\begin{cases}\bar v &= v/a\\ \bar x &= x/a\\ \bar t &= t/a^2\\ \bar K &= aK\end{cases}\qquad
(e)\begin{cases}\bar v &= bv\\ \bar x &= bx\\ \bar t &= bt\\ \bar D &= bD\end{cases}\tag{24}
$$

where $a, b \neq 0$ and $\alpha\delta-\beta\gamma=1$, and all parameters are *real*.

6.2 Defining system

The components χ, ξ, τ, η (functions of (v,x,t,u)) of the symmetry operator

$$\mathbf{Y} = \chi\,\partial_v + \xi\,\partial_x + \tau\,\partial_t + \eta\,\partial_u$$

satisfy the infinitesimal defining system

$$
\begin{gathered}
\tau_v = \tau_x = \tau_u = \xi_u = \chi_u = 0\\
\dot D(\,\partial_x + u\,\partial_v)(\chi - u\xi) - 2D(\,\partial_x + u\,\partial_v)\xi + D\,\partial_t\tau = 0\\
\big(\,\partial_t + \dot K(\,\partial_x + u\,\partial_v) - K\,\partial_v\big)(\chi - u\xi) + K\,\partial_t\tau - D(\,\partial_x + u\,\partial_v)^2(\chi - u\xi) = 0\\
\eta = (\,\partial_x + u\,\partial_v)(\chi - u\xi).
\end{gathered}\tag{25}
$$

6.3 Construction of invariant frame

Invariants and invariant frames for the full equivalence group will be constructed, by successively enlarging subgroups (24) (a), (a,b), . . . , (a,b,c,d,e). At each step the following elements are constructed:

- Scalar differential invariants.
- Invariant frame operators.
- Invariant vector field components.

- Structure relations of the frame.
- Auxiliary system in terms of frame.
- Defining system for symmetries in terms of frame.

(a) Common translation symmetries

The translations (24a) constitute the common symmetries of the system (21). The invariant scalars are u, D, K. The coordinate frame $\partial_v,\ \partial_x,\ \partial_t,\ \partial_u$ is invariant, as are the infinitesimals $\chi,\ \xi,\ \tau,\ \eta$. The structure relations are trivial. The scalars obey

$$\begin{array}{llll} \partial_v u = \partial_x u & = \partial_t u & = 0, & \qquad \partial_u u = 1 \\ \partial_v K = \partial_x K & = \partial_t K & = 0, & \\ \partial_v D = \partial_x D & = \partial_t D & = 0. & \end{array} \tag{26}$$

Defining system (25) is already in invariant form; this amounts to saying that its coefficients do not depend on v, x, t.

(b) Galilean transformation

Now consider the group (24b). The scalar invariants are

$$u, \quad D, \quad J := \ddot{K} \tag{27}$$

An invariant frame Δ and infinitesimals θ are:

$$\begin{array}{ll} \Delta_1 = \partial_v & \theta^1 = \chi - (u\dot{K} - K)\tau \\ \Delta_2 = \partial_x & \theta^2 = \xi - \dot{K}\tau \\ \Delta_3 = \partial_t - K\,\partial_v + \dot{K}(\,\partial_x + u\,\partial_v) & \theta^3 = \tau \\ \Delta_4 = \partial_u & \theta^4 = \eta. \end{array} \tag{28}$$

The structure relations are trivial apart from

$$[\Delta_3, \Delta_4] = -uJ\Delta_1 \tag{29}$$

System (6) implies that the scalars satisfy

$$\begin{array}{llll} u_{,1} = u_{,2} & = u_{,3} & = 0, & \qquad u_{,4} = 1 \\ K_{,1} = K_{,2} & = K_{,3} & = 0, & \\ D_{,1} = D_{,2} & = D_{,3} & = 0 & \end{array} \tag{30}$$

Finally, defining system (25) becomes

$$\begin{gathered}\theta^3_{,1} = \theta^3_{,2} = \theta^3_{,4} = 0\\ \theta^2_{,4} = -J\theta^3, \qquad \theta^1_{,4} = -uJ\theta^3\\ \dot D(\Delta_2 + u\Delta_1)(\theta^1 - u\theta^2) - 2D(\Delta_2 + u\Delta_1)\theta^2 + D\Delta_3\theta^3 = 0\\ \Delta_3(\theta^1 - u\theta^2) - D(\Delta_2 + u\Delta_1)^2(\theta^1 - u\theta^2) = 0\\ \theta^4 = (\Delta_2 + u\Delta_1)(\theta^1 - u\theta^2)\end{gathered} \tag{31}$$

where dot notation $\Delta_4 D = \dot D$ etc. is retained, since $\Delta_4 = \partial_u$.

(c) $\boldsymbol{SL}_2$ subgroup

The group (24c) acts on the scalar invariants (27) of (24a,b) by

$$u' = \frac{\alpha u + \beta}{\gamma u + \delta}, \qquad D' = (\gamma u + \delta)^2 D, \qquad J' = (\gamma u + \delta)^3 J,$$

and on the frame Δ (28) by

$$\Delta_1' = \delta\Delta_1 - \gamma\Delta_2, \qquad \Delta_2' = -\beta\Delta_1 + \alpha\Delta_2, \qquad \Delta_3' = \Delta_3, \qquad \Delta_4' = (\gamma u + \delta)^2\Delta_4.$$

From this, we find scalar invariants

$$L := \frac{D\ddot D - 3/2\dot D^2}{D^4}, \qquad I := |J| D^{-3/2}, \tag{32}$$

and invariant frame Λ and infinitesimals λ:

$$\begin{aligned} &\Lambda_1 = \pi D^{1/2}(\Delta_2 + u\Delta_1) && \lambda^1 = -\tfrac{1}{2}\pi D^{-3/2}\big(\dot D(\theta^1 - u\theta^2) - 2D\theta^2\big)\\ &\Lambda_2 = \pi D^{-3/2}\big(2D\Delta_1 + \dot D(\Delta_2 + u\Delta_1)\big) && \lambda^2 = \tfrac{1}{2}\pi D^{1/2}(\theta^1 - u\theta^2)\\ &\Lambda_3 = \Delta_3 && \lambda^3 = \theta^3\\ &\Lambda_4 = \tfrac{1}{D}\Delta_4, && \lambda^4 = D\theta^4. \end{aligned} \tag{33}$$

The quantity π appearing throughout is a sign $\pm$, whose choice is discussed below.

The nontrivial structure relations of the frame Λ are

$$[\Lambda_1, \Lambda_4] = -\tfrac{1}{2}\Lambda_2, \qquad [\Lambda_2, \Lambda_4] = -L\Lambda_1, \qquad [\Lambda_3, \Lambda_4] = -I\Lambda_1 \tag{34}$$

System (30) shows that the invariants L, I are subject to

$$L_{,1} = L_{,2} = L_{,3} = 0, \qquad I_{,1} = I_{,2} = I_{,3} = 0. \tag{35}$$

Finally, defining system (31) becomes

$$\begin{array}{llll}
\lambda^3_{,1} = 0 & \lambda^1_{,1} = \frac{1}{2}\lambda^3_{,3} & \lambda^2_{,11} = \lambda^2_{,3} & \lambda^4 = 2\lambda^2_{,1} \\
\lambda^3_{,2} = 0 & & & \\
\lambda^3_{,4} = 0 & \lambda^1_{,4} = -L\lambda^2 - I\lambda^3 & \lambda^2_{,4} = -\frac{1}{2}\lambda^1 &
\end{array} \tag{36}$$

In this beautiful form only two terms have nonconstant coefficients, and the simplicity of structure of defining system (25) is revealed.

Consider the sign π which appeared above in (28). If $J \neq 0$, then π can be chosen as $\pi = \operatorname{sgn} J$, and the frame Λ is then invariant under the action of the whole SL_2 subgroup (24c). However, if $J = 0$ then Λ_1, Λ_2 (28) change sign under action of (24c), so that (28) is not invariant. This is because when $I = 0$ (pure diffusion) the transformation $x \mapsto -x$, $v \mapsto -v$ becomes a reflection *symmetry*. In this case we set $\pi = 1$.

(d) Scaling group—convection

The scaling group (24d) acts on Λ given in (32), (33) by

$$\begin{array}{ll}
\Lambda'_1 = a\Lambda_1 & L' = L \\
\Lambda'_2 = a\Lambda_2 & I' = aI \\
\Lambda'_3 = a^2\Lambda_3 & \\
\Lambda'_4 = \Lambda_4 &
\end{array}$$

There is now a case splitting, depending whether $I = 0$ or not.

Case a. $I \neq 0$.

The parameter a can be eliminated to give an invariant frame Γ and infinitesimals ζ:

$$\begin{array}{ll}
\Gamma_1 = I^{-1}\Lambda_1 & \zeta^1 = I\lambda^1 \\
\Gamma_2 = I^{-1}\Lambda_2 & \zeta^2 = I\lambda^2 \\
\Gamma_3 = I^{-2}\Lambda_3 & \zeta^3 = I^2\lambda^3 \\
\Gamma_4 = \quad\ \Lambda_4 & \zeta^4 = \lambda^4.
\end{array} \tag{37}$$

The scalar invariants are

$$L, \qquad M := I_{,4}/I \tag{38}$$

The structure relations of Γ are

$$[\Gamma_1, \Gamma_4] = -\tfrac{1}{2}\Gamma_2 + M\Gamma_1, \quad [\Gamma_2, \Gamma_4] = -L\Gamma_1 + M\Gamma_2, \quad [\Gamma_3, \Gamma_4] = -\Gamma_1 + 2M\Gamma_3 \tag{39}$$

From (35) the invariants L, M obey

$$L_{,1} = L_{,2} = L_{,3} = 0, \qquad M_{,1} = M_{,2} = M_{,3} = 0, \tag{40}$$

Finally, defining system (36) becomes

$$\begin{array}{llll} \zeta^3_{,1} = 0, & \zeta^1_{,1} = \frac{1}{2}\zeta^3_{,3}, & \zeta^2_{,11} = \zeta^2_{,3}, & \zeta^4 = 2\zeta^2_{,1} \\ \zeta^3_{,2} = 0, & & & \\ \zeta^3_{,4} = 2M\zeta^3 & \zeta^1_{,4} = M\zeta^1 - L\zeta^2 - \zeta^3, & \zeta^2_{,4} = -\frac{1}{2}\zeta^1 + M\zeta^2 & \end{array} \tag{41}$$

Case b. $I = 0$.

The condition $I = 0$ is equivalent to $J = 0$, that is, $\dddot{K} = 0$. This case is equivalent to a pure diffusion equation $K = 0$. Here division by I cannot be effected, so the group parameter a cannot be eliminated: the Boltzmann scaling operator has become a symmetry. An invariant L is available, but there is no invariant frame. Note that this case also inherits the symmetry $x \mapsto -x$, $v \mapsto -v$ from the SL_2 group (24c): this is due to our failure to eliminate the sign π.

(e) Scaling group—diffusion

Finally the scaling group (24e) acts on Λ, L, I as follows:

$$\begin{array}{ll} \Lambda'_1 = b^{-1/2}\Lambda_1 & L' = b^{-2}L \\ \Lambda'_2 = b^{-3/2}\Lambda_2 & I' = b^{-3/2}I \\ \Lambda'_3 = b^{-1}\Lambda_3 & \\ \Lambda'_4 = b^{-1}\Lambda_4 & \end{array} \tag{42}$$

Now consider the branches $I \neq 0$ and $I = 0$ from above.

Case a. $I \neq 0$.

The action of (24e) on Γ, L, M is:

$$\begin{array}{ll} \Gamma'_1 = b\Gamma_1 & L' = b^{-2}L \\ \Gamma'_2 = \Gamma_2 & M' = b^{-1}M \\ \Gamma'_3 = b^2\Gamma_3 & \\ \Gamma'_4 = b^{-1}\Gamma_4 & \end{array}$$

The calculation splits into two subcases, depending whether L vanishes.

Case aa. $I \neq 0$, $L \neq 0$.

Using L, the parameter b can be eliminated, yielding invariant frame Σ and infinitesimals β

$$\begin{aligned} \Sigma_1 &= |L|^{1/2}\Gamma_1 & \beta^1 &= |L|^{-1/2}\zeta^1 \\ \Sigma_2 &= \Gamma_2 & \beta^2 &= \zeta^2 \\ \Sigma_3 &= L\Gamma_3 & \beta^3 &= L^{-1}\zeta^3 \\ \Sigma_4 &= |L|^{-1/2}\Gamma_4 & \beta^4 &= |L|^{1/2}\zeta^4. \end{aligned} \tag{43}$$

The invariants of the group action are

$$P := |L|^{-3/2}\Gamma_4 L, \qquad Q := M|L|^{-1/2}, \qquad \sigma := \operatorname{sgn} L. \tag{44}$$

The sign σ is genuinely invariant, so long as transformations are real-valued (or complex-valued). The structure relations of the frame Σ are

$$\begin{aligned} [\Sigma_1, \Sigma_4] = \tfrac{1}{2}(2Q - \sigma P)\Sigma_1 - \tfrac{1}{2}\Sigma_2, \quad & [\Sigma_2, \Sigma_4] = -\sigma\Sigma_1 + Q\Sigma_2, \\ & [\Sigma_3, \Sigma_4] = -\sigma\Sigma_1 + (2Q - \sigma P)\Sigma_3 \end{aligned} \tag{45}$$

From (40), the invariants P, Q are bound by

$$P_{,1} = P_{,2} = P_{,3} = 0, \qquad Q_{,1} = Q_{,2} = Q_{,3} = 0. \tag{46}$$

Finally, defining system (41) becomes

$$\begin{aligned} \beta^3_{,1} &= 0, & \beta^1_{,1} &= \tfrac{1}{2}\beta^3_{,3}, \quad \beta^2_{,11} = \sigma\beta^2_{,3}, & \beta^4 &= 2\beta^2_{,1} \\ \beta^3_{,2} &= 0, & & \qquad \beta^2_{,4} = -\tfrac{1}{2}\beta^1 + Q\beta^2 & & \\ \beta^3_{,4} &= (2Q - \sigma P)\beta^3 & \beta^1_{,4} &= \tfrac{1}{2}(2Q - \sigma P)\beta^1 - \sigma\beta^2 - \sigma\beta^3 & & \end{aligned} \tag{47}$$

Case ab. $I \neq 0$, $L = 0$.

Here L cannot be used to eliminate the parameter b. There is a case splitting on M.

Case aba. $I \neq 0$, $L = 0$, $M \neq 0$.

Here M can be used to eliminate the group parameter b. The scalar invariants are

$$R := M^{-2}\Gamma_4 M, \qquad S := LM^{-2}. \tag{48}$$

An invariant frame Ξ and infinitesimals ξ are:

$$\begin{aligned} \Xi_1 &= M\Gamma_1 & \xi^1 &= M^{-1}\zeta^1 \\ \Xi_2 &= \Gamma_2 & \xi^2 &= \zeta^2 \\ \Xi_3 &= M^2\Gamma_3 & \xi^3 &= M^{-2}\zeta^3 \\ \Xi_4 &= M^{-1}\Gamma_4 & \xi^4 &= M\zeta^4. \end{aligned} \tag{49}$$

The quantities R, S, Ξ are well-defined whenever $M \neq 0$ (regardless of L). Here $L = 0$, and hence $S = 0$. The structure relations of Ξ are

$$[\Xi_1, \Xi_4] = -\tfrac{1}{2}\Xi_2 + (1-R)\Xi_1 \qquad [\Xi_2, \Xi_4] = \Xi_2$$
$$[\Xi_3, \Xi_4] = -\Xi_1 + 2(1-R)\Xi_3 \tag{50}$$

From (40) the invariant R is bound by

$$R_{,1} = R_{,2} = R_{,3} = 0. \tag{51}$$

Finally, the defining system (41) becomes

$$\begin{array}{llll} \xi^3_{,1} = 0, & \xi^1_{,1} = \tfrac{1}{2}\xi^3_{,3}, & \xi^2_{,11} = \xi^2_{,3}, & \xi^4 = 2\xi^2_{,1} \\ \xi^3_{,2} = 0, & & & \\ \xi^3_{,4} = 2(1-R)\xi^3 & \xi^1_{,4} = (1-R)\xi^1 - \xi^3, & \xi^2_{,4} = -\tfrac{1}{2}\xi^1 + \xi^2 & \end{array} \tag{52}$$

Case abb. $I \neq 0$, $L = 0$, $M = 0$.

Here there is no way to eliminate the group parameter b, and we retain the frame Γ. This singular case is again associated with equivalence transformations moving into the symmetry group. Conditions $L = 0$, $M = 0$, $I \neq 0$ are

$$D\ddot{D} - \tfrac{3}{2}\dot{D}^2 = 0, \qquad (\ddot{K}D^{-3/2})^{\cdot} = 0, \qquad \ddot{K} \neq 0$$

which lead to

$$D(u) = (eu+f)^{-2}, \qquad K(u) = \frac{au^2 + bu + c}{eu + f}$$

where at least one of e, f is nonvanishing, and $eu+f$ does not divide $au^2 + bu + c$. These are equations including Burgers' equation and a system studied by Fokas and Yortsos [9]. It is interesting that the frame calculations pick this out as a singular case even though the linearizing transformation taking Burgers' to the heat equation is not in the equivalence group, and hence not detected.

Case b. $I = 0$.

The action of the scaling group (24e) on the frame Λ is given by (42). With $I = 0$, the calculation splits on L.

Case ba. $I = 0$, $L \neq 0$.

Here L can be used to eliminate the parameter b. The scalar invariants of the group action are

$$P := |L|^{-3/2}\Lambda_4 L, \qquad \sigma := \operatorname{sgn} L. \tag{53}$$

An invariant frame Ω and infinitesimals ω are:

$$\begin{aligned} \Omega_1 &= |L|^{-1/4}\Lambda_1 & \omega^1 &= |L|^{1/4}\lambda^1 \\ \Omega_2 &= |L|^{-3/4}\Lambda_2 & \omega^2 &= |L|^{3/4}\lambda^2 \\ \Omega_3 &= |L|^{-1/2}\Lambda_3 & \omega^3 &= |L|^{1/2}\lambda^3 \\ \Omega_4 &= |L|^{-1/2}\Lambda_4 & \omega^4 &= |L|^{1/2}\lambda^4. \end{aligned} \tag{54}$$

This P is the same as (44), merely rewritten in new notation. The structure relations of the frame Ω are

$$\begin{aligned} [\Omega_1, \Omega_4] &= \tfrac{1}{4}\sigma P\Omega_1 - \tfrac{1}{2}\Omega_2 \qquad & [\Omega_2, \Omega_4] &= -\sigma\Omega_1 + \tfrac{3}{4}\sigma P\Omega_2, \\ & & [\Omega_3, \Omega_4] &= \tfrac{1}{2}\sigma P\Omega_3. \end{aligned} \tag{55}$$

From (35) the invariant P satisfies

$$P_{,1} = P_{,2} = P_{,3} = 0 \tag{56}$$

Finally, determining system (36) becomes

$$\begin{aligned} &\omega^3_{,1} = 0 \qquad \omega^1_{,1} = \tfrac{1}{2}\omega^3_{,3} \qquad \omega^2_{,11} = \omega^2_{,3} \qquad \omega^4 = 2\omega^2_{,1} \\ &\omega^3_{,2} = 0 \\ &\omega^3_{,4} = \tfrac{1}{2}\sigma P\omega^3 \quad \omega^1_{,4} = \tfrac{1}{4}\sigma P\omega^1 - \sigma\omega^2 \quad \omega^2_{,4} = -\tfrac{1}{2}\omega^1 + \tfrac{3}{4}\sigma P\omega^2 \end{aligned} \tag{57}$$

Case bb. $I = 0$, $L = 0$.

Here there is no way to eliminate b, and frame Λ is retained. This singular case corresponds to D, K satisfying

$$D\ddot{D} - \tfrac{3}{2}\dot{D}^2 = 0, \qquad \ddot{K} = 0$$

or

$$D(u) = (eu + f)^{-2}, \qquad K(u) = au + b$$

where at least one of e, f is nonvanishing. These equations include the system studied in [36, 3] and found to be equivalent to the linear heat system. In this case the linearizing transformation is in the equivalence group, so it is expected that this case should be picked out as singular.

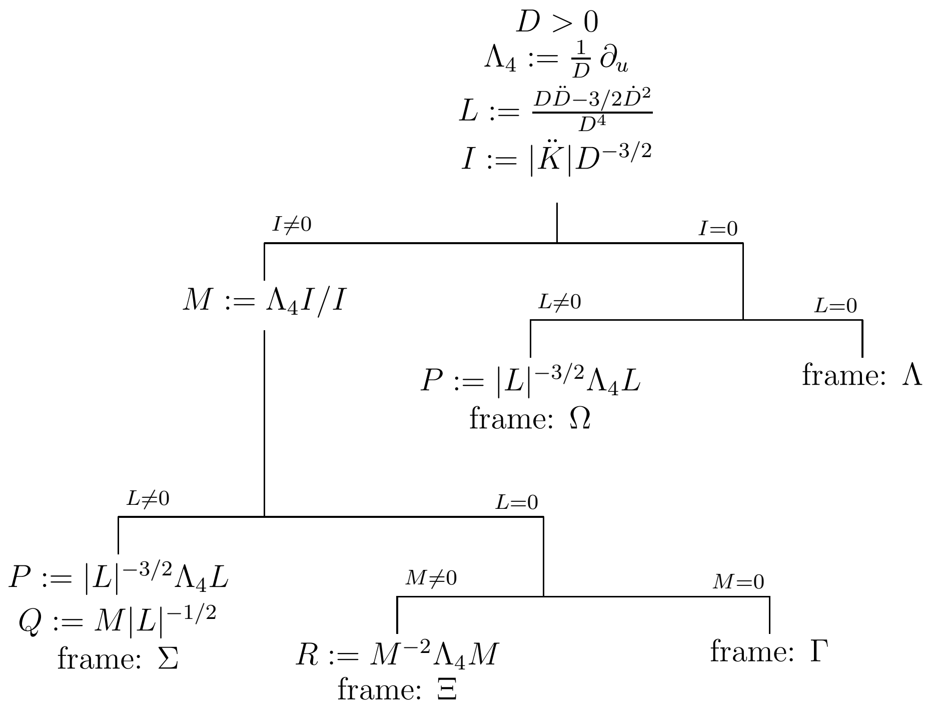

Figure 1: Preliminary classification tree for potential diffusion convection system (21). Branchings are on the basis of whether or not particular frames exist.

6.4 Completion to non-commutative Riquier Basis form

So far we have the incomplete classification tree shown in Figure 1.

For each of the five leaves of the tree above we now complete the determining system to a non-commutative Riquier Basis. This gives rise to a further hierarchy of branchings. Note that three common translation symmetries are always present, and we do not present results for any branch with only a three-dimensional solution space.

Case aa. $I \neq 0$, $L \neq 0$.

The defining system is (47), referred to frame Σ. The frame Riquier method gives case splittings which show that for symmetry beyond the minimal translations, it is necessary that $P_{,4} = Q_{,4} = 0$, implying that P, Q are constants. In that case, the system reduces to the non-commutative Riquier

Basis

$$\begin{array}{llll}
\beta^3_{,1} = 0 & \beta^2_{,11} = 0 & \beta^1_{,1} = -(2Q-\sigma P)\beta^2_{,1} & \\
\beta^3_{,2} = 0 & \beta^2_{,2} = -2Q\beta^2_{,1} & \beta^1_{,2} = 2\sigma\beta^2_{,1} & \beta^4 = 2\beta^2_{,1} \\
\beta^3_{,3} = -2(2Q-\sigma P)\beta^2_{,1} & \beta^2_{,3} = 0 & \beta^1_{,3} = 2\sigma\beta^2_{,1} & \\
\beta^3_{,4} = (2Q-\sigma P)\beta^3 & \beta^2_{,4} = -\frac{1}{2}\beta^1 + Q\beta^2 & \beta^1_{,4} = \frac{1}{2}(2Q-\sigma P)\beta^1 - \sigma\beta^2 - \sigma\beta^3 &
\end{array} \tag{58}$$

The four parametric quantities β^1, β^2, β^3, $\beta^2_{,1}$ give a four-parameter symmetry group.

Application of the method (Section 5) for finding structure constants gives a Lie algebra of symmetry operators $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$, $\mathbf{Y}_4$ with commutation relations

$$[\mathbf{Y}_1,\mathbf{Y}_4] = -(2Q-\sigma P)\mathbf{Y}_1 + \mathbf{Y}_2 \qquad [\mathbf{Y}_2,\mathbf{Y}_4] = 2\sigma\mathbf{Y}_1 - 2Q\mathbf{Y}_2$$
$$[\mathbf{Y}_3,\mathbf{Y}_4] = 2\sigma\mathbf{Y}_1 - 2(2Q-\sigma P)\mathbf{Y}_2$$

Case aba. $I \neq 0$, $L = 0$, $M \neq 0$.

Applying the frame Riquier method to system (52) with frame Ξ (49) shows that additional symmetry only arises only if $R_{,4} = 0$, in which case the non-commutative Riquier Basis is:

$$\begin{array}{lll}
\xi^3_{,1} = 0 & \xi^2_{,1} = -\frac{1}{2}\xi^2_{,2} & \xi^1_{,1} = (1-R)\xi^2_{,2} \\
\xi^3_{,2} = 0 & \xi^2_{,22} = 0 & \xi^1_{,2} = 0 \\
\xi^3_{,3} = 2(1-R)\xi^2_{,2} & \xi^2_{,3} = 0 & \xi^1_{,3} = -\xi^2_{,2} \\
\xi^3_{,4} = 2(1-R)\xi^3 & \xi^2_{,4} = -\frac{1}{2}\xi^1 + \xi^2 & \xi^1_{,4} = (1-R)\xi^1 - \xi^3 \\
 & & \xi^4 = -\xi^2_{,2}
\end{array} \tag{59}$$

The parametric quantities ξ^1, ξ^2, ξ^3, $\xi^2_{,2}$ give a four-dimensional symmetry algebra with commutation relations

$$[\mathbf{Y}_1,\mathbf{Y}_4] = (1-R)\mathbf{Y}_1 - \tfrac{1}{2}\mathbf{Y}_2, \quad [\mathbf{Y}_2,\mathbf{Y}_4] = \mathbf{Y}_2, \quad [\mathbf{Y}_3,\mathbf{Y}_4] = -\mathbf{Y}_1 + 2(1-R)\mathbf{Y}_3$$

Case abb. $I \neq 0$, $L = 0$, $M = 0$.

No further case splittings arise for this case, which includes Burgers' equation and Fokas-Yortsos' equation [9]. The Cole-Hopf transformation connects the potential system to the linear heat system, and the non-commutative Riquier Basis has infinitely many parametric derivatives. It is not reproduced here.

Case ba. $I = 0$, $L \neq 0$.

The frame Riquier method applied to defining system (57) yields a case splitting on $P_{,4}$. If $P_{,4} \neq 0$ the non-commutative Riquier Basis is

$$\begin{array}{lll} \omega^3_{,1} = 0 & \omega^2_{,1} = 0 & \omega^1_{,1} = \omega^2_{,2} \\ \omega^3_{,2} = 0 & \omega^2_{,22} = 0 & \omega^1_{,2} = 0 \\ \omega^3_{,3} = 2\omega^2_{,2} & \omega^2_{,3} = 0 & \omega^1_{,3} = 0 \\ \omega^3_{,4} = \frac{1}{2}\sigma P\omega^3 & \omega^2_{,4} = -\frac{1}{2}\omega^1 + \frac{3}{4}\sigma P\omega^2 & \omega^1_{,4} = \frac{1}{4}\sigma P\omega^1 - \sigma\omega^2 \\ & & \omega^4 = 0 \end{array} \tag{60}$$

The parametric derivatives ω^1, ω^2, ω^3, $\omega^2_{,2}$ give the four-dimensional symmetry algebra common to all diffusion potential systems. The commutation relations are

$$[\mathbf{Y}_1, \mathbf{Y}_4] = \mathbf{Y}_1, \qquad [\mathbf{Y}_2, \mathbf{Y}_4] = \mathbf{Y}_2, \qquad [\mathbf{Y}_3, \mathbf{Y}_4] = \mathbf{Y}_3.$$

If $P_{,4} = 0$, we obtain a non-commutative Riquier Basis

$$\begin{array}{lll} \omega^3_{,1} = 0 & \omega^2_{,11} = 0 & \omega^1_{,1} = \sigma P\omega^2_{,1} + \omega^2_{,2} \\ \omega^3_{,2} = 0 & \omega^2_{,12} = 0 & \omega^1_{,2} = 2\sigma\omega^2_{,1} \\ \omega^3_{,3} = 2\sigma P\omega^2_{,1} + 2\omega^2_{,2} & \omega^2_{,22} = 0 & \omega^1_{,3} = 0 \\ \omega^3_{,4} = \frac{1}{2}\sigma P\omega^3 & \omega^2_{,3} = 0 & \omega^1_{,4} = \frac{1}{4}\sigma P\omega^1 - \sigma\omega^2 \\ & \omega^2_{,4} = -\frac{1}{2}\omega^1 + \frac{3}{4}\sigma P\omega^2 & \omega^4 = 2\omega^2_{,1} \end{array} \tag{61}$$

The parametric quantities ω^1, ω^2, ω^3, $\omega^2_{,1}$, $\omega^2_{,2}$ give a five-dimensional symmetry algebra, with structure relations

$$\begin{array}{lll} [\mathbf{Y}_1, \mathbf{Y}_4] = \frac{1}{2}\mathbf{Y}_1 & [\mathbf{Y}_1, \mathbf{Y}_5] = \mathbf{Y}_2 & [\mathbf{Y}_2, \mathbf{Y}_4] = \frac{1}{2}\mathbf{Y}_2 \\ [\mathbf{Y}_2, \mathbf{Y}_5] = 2\sigma\mathbf{Y}_1 - \sigma P\mathbf{Y}_2 & [\mathbf{Y}_3, \mathbf{Y}_4] = \mathbf{Y}_3 & \end{array}$$

Case bb. $I = 0$, $L = 0$

There is no further case splitting. The non-commutative Riquier Basis for (36) is

$$\begin{array}{lll} \lambda^3_{,1} = 0 & \lambda^2_{,4} = -\frac{1}{2}\lambda^1 & \lambda^1_{,1} = \frac{1}{2}\lambda^3_{,3} \\ \lambda^3_{,2} = 0 & \lambda^2_{,11} = \lambda^2_{,3} & \lambda^1_{,2} = 0 \\ \lambda^3_{,4} = 0 & \lambda^2_{,12} = -\frac{1}{2}\lambda^1_{,3} & \lambda^1_{,4} = 0 \\ \lambda^3_{,33} = 0 & \lambda^2_{,22} = \frac{1}{2}\lambda^1_{,3} & \lambda^1_{,33} = 0 \\ & \lambda^2_{,23} = 0 & \lambda^4 = 2\lambda^2_{,1} \end{array} \tag{62}$$

There are infinitely many parametric quantities: λ^1, λ^3, $\lambda^1_{,3}$, $\lambda^2_{,2}$, $\lambda^3_{,3}$; and the sequences λ^2, $\lambda^2_{,3}$, $\lambda^2_{,33}$, ... and $\lambda^2_{,1}$, $\lambda^2_{,13}$, $\lambda^2_{,133}$, ... The symmetry algebra is therefore infinite-dimensional: this subcase consists of equations which can be mapped to the heat equation by an equivalence transformation, so its symmetry properties can be regarded as known.

6.5 Summary of classification

The calculations of this section yield the classification tree shown in Figure 2. In this compact diagram is present all the information required to decide the symmetry properties of a diffusion convection potential system. The elegance of the result is apparent when compared with the classifying equations produced by the 'raw' version of Riquier–Janet [28].

In Figure 2, all the branchings are (by construction) invariant under the action of the equivalence group. Hence two equations connected by an equivalence transformation always occur on the same branch. This greatly cuts down on spurious branchings. Note that equations occurring on different branches of the tree could be equivalent with respect to a transformation *not* in the group (24). Indeed Burgers' and linear heat equations occur on different branches, yet are connected by the Cole-Hopf transformation.

7 Linear Hyperbolic Equations

Consider the class of second order linear hyperbolic equations in characteristic coordinates:

$$\partial_{xy}z + A\,\partial_x z + B\,\partial_y z + Cz = 0, \tag{63}$$

with A, B, C functions of (x, y). We give a complete symmetry classification for these equations, completing the discussion by Ovsiannikov [25, §9] (see also Lie [16] and Vranceanu [38]).

7.1 Equivalence group

The equivalence transformations for (63) are

$$(a)\begin{cases} x' = x \\ y' = y \\ z' = w(x,y)z \end{cases} \qquad (b)\begin{cases} x' = f(x) \\ y' = g(y) \\ z' = z \end{cases} \qquad (c)\begin{cases} x' = y \\ y' = x \\ z' = z \end{cases}$$

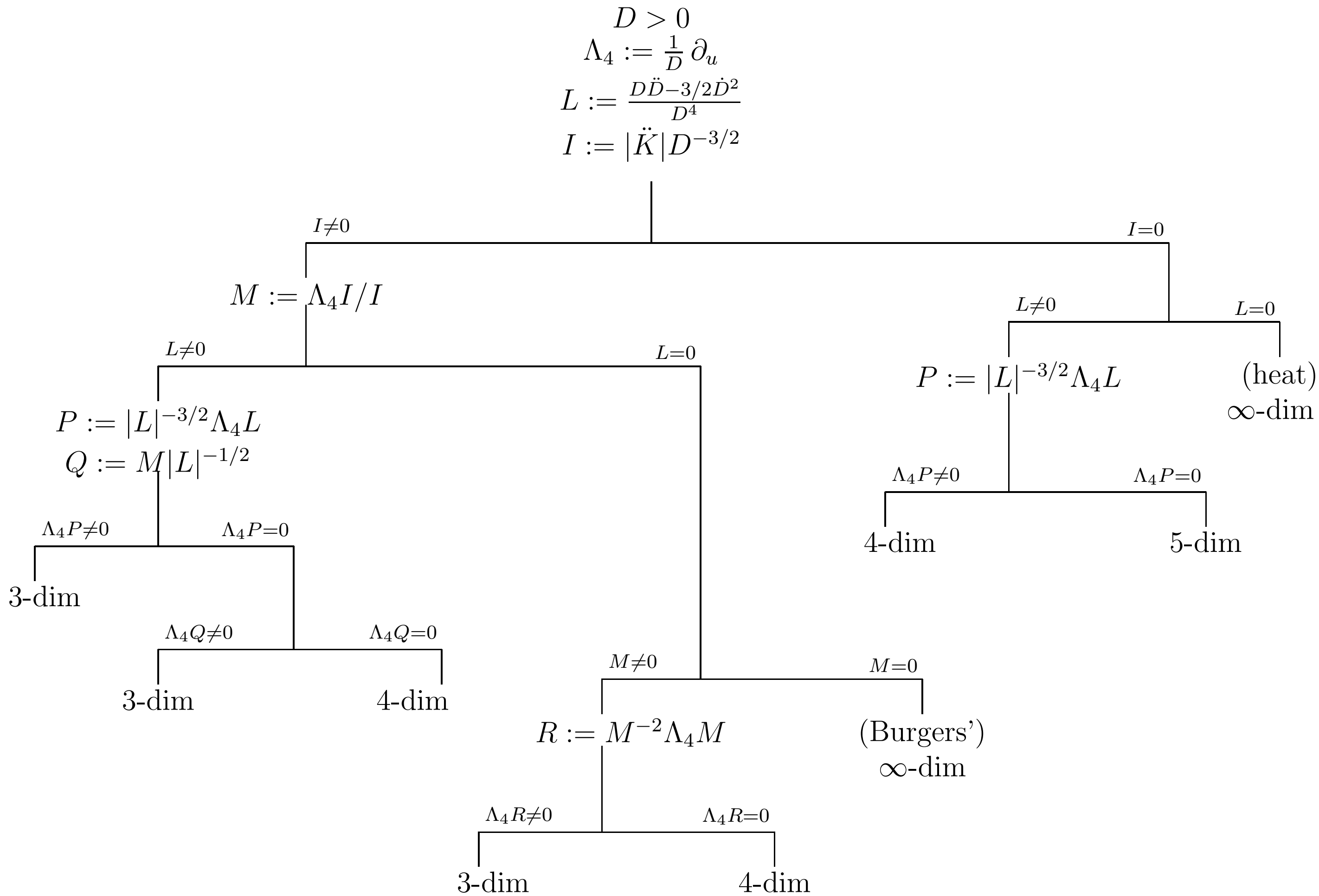

Figure 2: Complete symmetry classification tree for potential diffusion convection system (21).

where (a) is a normal subgroup of (a,b). We make only *ad hoc* use of the reflection (c). The effect of these transformations on the arbitrary elements is

$$
\text{(a)} \begin{cases} A' = A - \dfrac{w_x}{w} \\ B' = B - \dfrac{w_y}{w} \\ C' = C - A\dfrac{w_x}{w} - B\dfrac{w_y}{w} - \dfrac{w_{xy}}{w} \end{cases}
\qquad
\text{(b)} \begin{cases} A' = \dfrac{1}{\dot g}A \\ B' = \dfrac{1}{\dot f}B \\ C' = \dfrac{1}{\dot f \dot g}C \end{cases}
\qquad
\text{(c)} \begin{cases} A' = B \\ B' = A \\ C' = C \end{cases}
$$

For transformations (a), the scalar invariants are x, y and the Laplace invariants

$$h := A_x + AB - C, \qquad k := B_y + AB - C$$

and their derivatives.

7.2 Symmetry operator

A symmetry operator for (63) is sought in the form

$$\mathbf{X} = \xi\,\partial_x + \eta\,\partial_y + \sigma z\,\partial_z$$

where ξ, η, σ depend on (x, y) only. An invariant frame and infinitesimals for (a) are given by

$$
\begin{array}{ll}
\partial_x - Bz\,\partial_z & \xi \\
\partial_y - Az\,\partial_z & \eta \\
z\,\partial_z & \zeta := \sigma + B\xi + A\eta
\end{array}
$$

In terms of these, the defining system is conveniently written [25, §9.4] as:

$$
\begin{array}{rr}
\xi_y = 0, & \eta_x = 0 \\
(h\xi)_x + (h\eta)_y = 0 & (k\xi)_x + (k\eta)_y = 0 \\
\zeta_x = (h-k)\eta & \zeta_y = (k-h)\xi
\end{array}
\tag{64}
$$

Note that compatibility of the last two equations is satisfied as a consequence of the other equations. From now on we consider only the first 4 equations, and ignore the z-components.

The action of transformations (b) on h, k is

$$h' = \frac{1}{\dot{f}\dot{g}}h, \qquad k' = \frac{1}{\dot{f}\dot{g}}k$$

The conditions $h = 0$, $k = 0$ are invariant, and specify the class of the wave equation $z_{xy} = 0$. If $h = 0$, $k \neq 0$, transformations (c) effect the interchange $h \leftrightarrow k$ and enable us to assume without loss of generality that $h \neq 0$.

7.3 Scalar invariants

Some properties of the invariants of transformations (a,b) are easily verified:

Proposition 7.1. *Suppose $h \neq 0$. Then:*

(i) The quantities

$$p := k/h \qquad\qquad q := \frac{h_{xy}}{h^2} - \frac{h_x h_y}{h^3} = \frac{1}{h}\partial_{xy} \ln h$$

are scalar invariants.

(ii) The nonlinear operations

$$\mathcal{N}[J] := \frac{\partial_x J\, \partial_y J}{h} \qquad\qquad \mathcal{S}[J] := \frac{\partial_{xy} J}{h} \tag{65}$$

map an invariant J to another invariant.

We make the notations

$$n := \mathcal{N}[p], \qquad s := \mathcal{S}[p].$$

Further invariants will be defined as needed.

7.4 Analysis of defining system

From the defining system (64), elementary manipulations give

$$\xi_y = 0 \qquad\qquad \eta_x = 0 \tag{66a}$$

$$\xi_x = -\eta_y - \frac{h_x}{h}\xi - \frac{h_y}{h}\eta \tag{66b}$$

$$p_x\xi + p_y\eta = 0 \tag{66c}$$

Compatibility of the ξ_x, ξ_y equations gives

$$\eta_{yy} = -\frac{h_y}{h}\eta_y - \left(\frac{h_y}{h}\right)_y \eta - hq\xi \tag{67}$$

and compatibility of this with the η_x equation gives

$$q_x\xi + q_y\eta = 0 \tag{68}$$

A useful fact is the following:

Proposition 7.2. *Let J be an invariant. Suppose the defining system includes the equation*

$$J_x\xi + J_y\eta = 0$$

Then it follows that

$$(\mathcal{N}J)_x\xi + (\mathcal{N}J)_y\eta = 0$$

Proof. Differentiation and simplification modulo (66) shows that

$$\begin{aligned} J_x\eta_y - (J_{xy} - \frac{h_y}{h})\eta - (J_{xx} - \frac{h_x}{h}J_x)\xi = 0 \\ J_y\eta_y + J_{yy}\eta + J_{xy}\xi = 0 \end{aligned} \tag{69}$$

and eliminating the η_y terms yields the result. □

The basic symmetry properties [25, §9] of the equation are well-known, and easily worked out:

Proposition 7.3. *(i) The symmetry group is of dimension ≤ 3.*

(ii) If $p_xq_y - q_xp_y \neq 0$, the symmetry group is trivial.

(iii) The symmetry group is of dimension 3 iff

$$p_x = p_y = q_x = q_y = 0$$

(iv) If condition (iii) is not satisfied, the group is of dimension ≤ 1.

Proof. (i) This follows by counting the non-leading derivatives ξ, η, η_y in the system (66) and (67).

(ii) If the determinant is nonzero, then (66c) and (68) immediately give $\xi = \eta = 0$.

(iii) If p_x, p_y, q_x, q_y vanish, then the defining system (66a), (66b), (67) is a non-commutative Riquier Basis; the three parametric derivatives ξ, η, η_y show that the Lie algebra is of dimension 3.

(iv) If p_x or p_y is nonzero, (66c) combined with (69) would give an equation with η_y as leading derivative, reducing the count of non-leading derivatives to 2 (ξ, η). Equation (66c) then reduces the count to 1. The same argument applied to (67) gives the same result if q_x or q_y is nonzero. This establishes (iv) and the 'only if' part of (iii).

□

7.5 Frame calculation

To complete the symmetry classification, the cases admitting a 1-dimensional symmetry group must be distinguished. Details are given for one subcase only; the others are similar.

First suppose that $p_x \neq 0$. Introduce an invariant frame such that

$$\xi\,\partial_x + \eta\,\partial_y = \theta^1\Delta_1 + \theta^2\Delta_2$$

where

$$\Delta_1 = \frac{1}{p_x}\partial_x \qquad \theta^1 = p_x\xi$$

$$\Delta_2 = \frac{p_x}{h}\partial_y \qquad \theta^2 = \frac{h}{p_x}\eta$$

The structure relations of the frame are

$$[\Delta_1, \Delta_2] = s\Delta_1 + r\Gamma_1$$

where

$$r := \frac{1}{p_x}\Big(\frac{p_{xx}}{p_x} - \frac{h_x}{h}\Big)$$

is an invariant. We note the following syzygy relationships:

$$p_{,1} = 1, \qquad p_{,2} = n, \qquad n_{,1} = s + nr$$

In terms of this frame, the defining system is

$$\theta^1 + n\theta^2 = 0, \qquad \theta^1_{,2} - s\theta^1 = 0, \qquad \theta^2_{,1} + r\theta^2 = 0$$
$$\theta^1_{,1} + \theta^2_{,2} - r\theta^1 + s\theta^2 = 0 \tag{70}$$

Clearing θ^1 from the other equations and using the syzygies gives the equation $(n_{,2} - nn_{,1})\theta^2 = 0$. Hence the group is trivial unless

$$p_{,1}n_{,2} - p_{,2}n_{,1} = 0$$

If this condition is satisfied, the defining system is

$$\theta^1 = -n\theta^2, \qquad \theta^2_{,1} = -r\theta^2 \qquad \theta^2_{,2} = -nr\theta^2 \tag{71}$$

Computing compatibility shows that the group is trivial unless

$$p_{,1}r_{,2} - p_{,2}r_{,1} = 0$$

Moreover, if this condition is satisfied, defining system (71) is in the form of a non-commutative Riquier Basis, with one parametric derivative θ^2. Hence the symmetry algebra is 1-dimensional.

Now suppose that $p_x = 0$ but $p_y \neq 0$. An invariant frame is

$$\Delta_1 = \frac{p_y}{h}\partial_x \qquad\qquad \theta^1 = \frac{h}{p_y}\xi$$

$$\Delta_2 = \frac{1}{p_y}\partial_y \qquad\qquad \theta^2 = p_y\eta$$

Another invariant appears:

$$t := \frac{1}{p_y}\Big(\frac{p_{yy}}{p_y} - \frac{h_y}{h}\Big),$$

and we note the relations

$$p_{,1} = 0, \qquad p_{,2} = 1, \qquad s = 0, \qquad t_{,1} = -q \tag{72}$$

The defining system is

$$\theta^2 = 0, \qquad \theta^1_{,1} = 0, \qquad \theta^1_{,2} = -t\theta^1 \tag{73}$$

and taking compatibility and using (72), it follows that the algebra is trivial if $q \neq 0$. If $q = 0$ then (73) is a non-commutative Riquier Basis, with one parametric derivative θ^1, and hence a 1-dimensional symmetry algebra.

If $p_x = p_y = 0$, similar considerations arise when $q_x \neq 0$ or $q_y \neq 0$: we do not reproduce the calculations here.

In the case p, q both constant, we work in the frame

$$\Delta_1 = \partial_x \qquad \theta^1 = \xi$$
$$\Delta_2 = \frac{1}{h}\partial_y \qquad \theta^2 = h\eta.$$

The defining system in non-commutative Riquier Basis form is

$$\begin{aligned} \theta^1_{,1} &= -\theta^2_{,2} - \frac{h_x}{h}\theta^1 & \theta^2_{,1} &= \frac{h_x}{h}\theta^2 \\ \theta^1_{,2} &= 0 & \theta^2_{,22} &= -q\theta^1 \end{aligned} \tag{74}$$

7.6 Commutation relations

Let (θ^1, θ^2) and (ϕ^1, ϕ^2) be two solutions of defining system (74). If

$$[\theta^1\Delta_1 + \theta^2\Delta_2, \phi^1\Delta_1 + \phi^2\Delta_2] = \psi^1\Delta_1 + \psi^2\Delta_2$$

then after simplification modulo (74), we find

$$\psi^1 = -(\theta^1\phi^2_{,2} - \phi^1\theta^2_{,2}), \qquad \psi^2 = \theta^2\phi^2_{,2} - \phi^2\theta^2_{,2}, \qquad \psi^2_{,2} = q(\theta^1\phi^2 - \theta^2\phi^1)$$

and hence the symmetry algebra has structure

$$[\mathbf{X}_1, \mathbf{X}_2] = q\mathbf{X}_3, \qquad [\mathbf{X}_1, \mathbf{X}_3] = -\mathbf{X}_1, \qquad [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_2.$$

The complete classification tree is shown in Figure 3.

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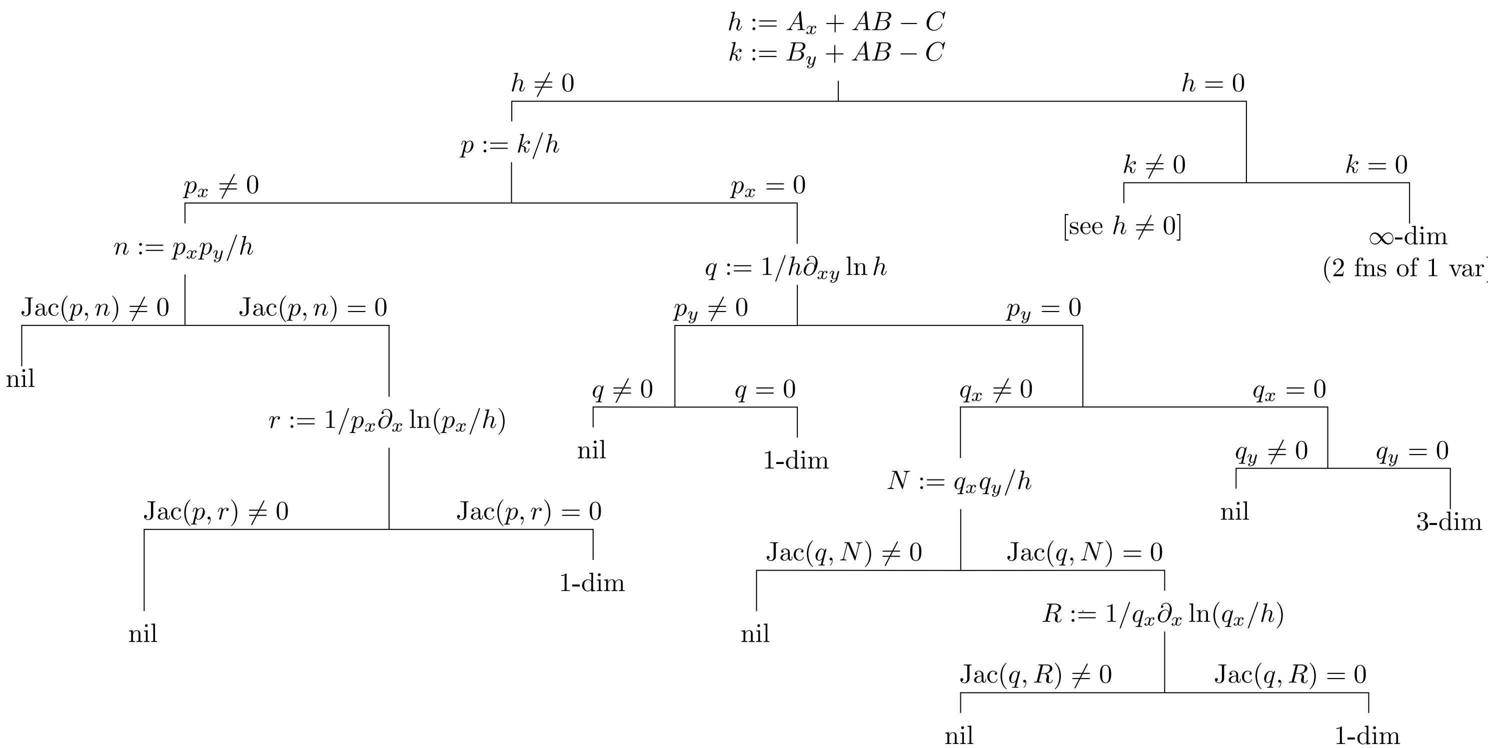

Figure 3: Classification tree for linear hyperbolic equations (63). The discrete transformation $x \leftrightarrow y$ is used to remap the case $h = 0$, $k \neq 0$ to $h \neq 0$, $k = 0$. Here $\text{Jac}(a, b) \equiv a_{,1} b_{,2} - a_{,2} b_{,1}$.

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